SS2 FURTHER MATHEMATICS LESSON NOTES

FIRST TERM

WEEK 1: SOLUTION TO QUADRATIC EQUATION

1. GENERAL FORM AND REVIEW

A quadratic equation is a polynomial equation of the second degree, with the general form: $ax^2 + bx + c = 0$ where a, b, and c are constant coefficients, and $a \neq 0$. The **roots** of the equation are the values of x that satisfy the equation.

---

2. METHODS OF SOLVING QUADRATIC EQUATIONS

2A. METHOD 1: FACTORIZATION

This method rewrites the quadratic expression as a product of two linear factors. It is the fastest method but only works if the roots are rational.

- **Worked Example 1:** Solve $6x^2 - x - 12 = 0$
    a. **Find ac:** $a = 6, c = -12 \Rightarrow ac = -72$.
    b. **Find factors of ac that sum to b:** We need two numbers that multiply to -72 and add to $b = -1$.
        - Factors: (1, -72), (2, -36), (3, -24), (4, -18), (6, -12), (8, -9).
        - The correct pair is **+8** and **-9**.
    c. **Split the middle term:** $6x^2 + 8x - 9x - 12 = 0$
    d. **Factor by grouping:** $(6x^2 + 8x) + (-9x - 12) = 0$ $2x(3x + 4) - 3(3x + 4) = 0$
    e. **Factor out the common bracket:** $(2x - 3)(3x + 4) = 0$
    f. **Solve for x:**
        - $2x - 3 = 0 \Rightarrow 2x = 3 \Rightarrow x = 3/2$

- $3x + 4 = 0 \Rightarrow 3x = -4 \Rightarrow x = -4/3$

- The roots are $x = 3/2$ or $x = -4/3$.

2B. METHOD 2: COMPLETING THE SQUARE

This method transforms the equation into a perfect square trinomial. It works for all quadratic equations.

- **Worked Example 2:** Solve $2x^2 - 6x - 1 = 0$
  a. **Move the constant:** $2x^2 - 6x = 1$
  b. **Divide by a (if $a \neq 1$):** $x^2 - 3x = 1/2$
  c. **Find $\left(\frac{b}{2}\right)^2$:** Take the new coefficient of x (which is -3), divide by 2, and square it.
     - $\left(\frac{-3}{2}\right)^2 = \frac{9}{4}$
  d. **Add this value to *both sides*:** $x^2 - 3x + \frac{9}{4} = \frac{1}{2} + \frac{9}{4}$
  e. **Factor the left (perfect square):** $\left(x - \frac{3}{2}\right)^2 = \frac{2}{4} + \frac{9}{4} \left(x - \frac{3}{2}\right)^2 = \frac{11}{4}$
  f. **Take the square root of both sides:** $x - \frac{3}{2} = \pm \sqrt{\frac{11}{4}} x - \frac{3}{2} = \pm \frac{\sqrt{11}}{2}$
  g. **Isolate x:** $x = \frac{3}{2} \pm \frac{\sqrt{11}}{2}$
  - The roots are $x = \frac{3+\sqrt{11}}{2}$ or $x = \frac{3-\sqrt{11}}{2}$.

2C. METHOD 3: THE QUADRATIC FORMULA

This formula is derived from completing the square and can solve *any* quadratic equation.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The **Discriminant ($\Delta = b^2 - 4ac$)** tells us the nature of the roots:

- If $\Delta > 0$: Two distinct real roots.

- If $\Delta = 0$: One real root (or two equal roots).
- If $\Delta < 0$: No real roots (two complex conjugate roots).
- **Worked Example 3:** Solve $3x^2 + 5x - 4 = 0$
    a. **Identify coefficients:** $a = 3, b = 5, c = -4$
    b. **Check discriminant:** $\Delta = 5^2 - 4(3)(-4) = 25 + 48 = 73$. Since $\Delta > 0$, we have two distinct real roots.
    c. **Substitute into the formula:** $x = \frac{-5 \pm \sqrt{73}}{2(3)}$
    d. **Simplify:** $x = \frac{-5 \pm \sqrt{73}}{6}$
    - The roots (which are irrational) are $x = \frac{-5+\sqrt{73}}{6}$ or $x = \frac{-5-\sqrt{73}}{6}$.

---

3. WORD PROBLEMS LEADING TO QUADRATIC EQUATIONS

- **Worked Example 4:** A rectangular garden is 4 meters longer than it is wide. If its area is 192 m^2, find the dimensions of the garden.
    a. **Define variables:**
        - Let the width be w meters.
        - Let the length be $l = (w + 4)$ meters.
    b. **Formulate the equation:**
        - Area = length × width
        - $192 = (w + 4) \times w$
    c. **Expand and set to zero:** $192 = w^2 + 4w\ w^2 + 4w - 192 = 0$
    d. **Solve:** We can use the formula or factorization. Let's try factorization.
        - Find factors of -192 that add to 4. (Hint: 12 and 16 are close).
        - The pair is **+16** and **-12**.
        - $(w + 16)(w - 12) = 0$
    e. **Find the roots:**
        - $w = -16$ or $w = 12$

f. **Interpret the result:**

- Since width (a distance) cannot be negative, we **reject** $w = -16$.
- The width is **12 meters**.
- The length is $w + 4 = 12 + 4 = 16$ **meters**.

- **Answer:** The garden is 16m by 12m.

EVALUATION

1. What is the "discriminant" of a quadratic equation, and what does it tell us if its value is negative?
2. Solve $x^2 - 13x + 40 = 0$ using the factorization method.
3. Solve $x^2 - 8x + 11 = 0$ using the method of completing the square.
4. Solve $5x^2 + x - 3 = 0$ using the quadratic formula.
5. The product of two consecutive positive odd integers is 143. Find the integers. (Hint: Let the integers be n and $n + 2$).

ASSIGNMENT

1. Derive the quadratic formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ from the general form $ax^2 + bx + c = 0$ using the "completing the square" method. Show every step.
2. Solve for x in $9x^2 - 12x + 4 = 0$. What do you notice about the root(s)? Check the discriminant and explain.
3. Find the values of k for which the equation $kx^2 - 3x + 1 = 0$ has two distinct real roots. (Hint: Set $\Delta > 0$).
4. Solve $3x^2 - 7x - 2 = 0$ using the quadratic formula, leaving your answer in surd form.
5. A projectile is fired. Its height h (in meters) after t (in seconds) is given by $h(t) = 30t - 5t^2$.
   - (a) At what times t will the projectile be at a height of 40 meters?
   - (b) How long does it take for the projectile to return to the ground?

WEEK 2: TANGENT AND NORMAL TO CURVE

1. MEANING OF TANGENT AND NORMAL

This topic is a direct and fundamental application of **Differential Calculus**. The core concept is that the **derivative (dy/dx)** of a function $y = f(x)$ represents the **gradient (slope) of the curve** at any given point x.

- **TANGENT:** A **tangent** is a straight line that "just touches" a curve at a single, specific point. It represents the *instantaneous rate of change* (the slope) of the curve at that exact point.
    - The gradient of the tangent line (m_T) is *equal* to the derivative of the curve at that point.
    - $m_T = \left(\frac{dy}{dx}\right)$ at (x_1, y_1)
- **NORMAL:** A **normal** is a straight line that is **perpendicular** to the tangent line at the *same point* of contact.
    - From coordinate geometry, we know that for two perpendicular lines, the product of their gradients is -1. ($m_1 \times m_2 = -1$).
    - Therefore, the gradient of the normal line (m_N) is the **negative reciprocal** of the tangent's gradient.
    - $m_N = -\frac{1}{m_T} = -\frac{1}{dy/dx}$

2. DERIVATION OF TANGENT AND NORMAL EQUATIONS

To find the equation of *any* straight line, we need two things:

1. A point on the line: (x_1, y_1)
2. The gradient of the line: m

The equation is given by the point-slope form: $y - y_1 = m(x - x_1)$

2A. STEPS TO FIND THE EQUATION OF THE TANGENT

1. **Given:** A curve $y = f(x)$ and a point x_1.
2. **Find the y-coordinate:** Substitute x_1 into the original function $f(x)$ to find y_1. This gives you your point (x_1, y_1).
3. **Find the derivative:** Find the derivative of the function, $dy/dx = f'(x)$.
4. **Find the tangent gradient (m_T):** Substitute x_1 into the *derivative* $f'(x)$ to find the gradient of the tangent at that point. $m_T = f'(x_1)$.
5. **Find the equation:** Substitute the point (x_1, y_1) and the gradient m_T into the line equation: $y - y_1 = m_T(x - x_1)$.

2B. STEPS TO FIND THE EQUATION OF THE NORMAL

1. **Complete Steps 1-4** from the tangent derivation. You now have the point (x_1, y_1) and the tangent gradient m_T.
2. **Find the normal gradient (m_N):** Invert the tangent gradient and change its sign. $m_N = -\frac{1}{m_T}$
3. **Find the equation:** Substitute the point (x_1, y_1) and the *normal* gradient m_N into the line equation: $y - y_1 = m_N(x - x_1)$.

---

3. WORKED EXAMPLES

- **Worked Example 1:** Find the equations of the tangent and the normal to the curve $y = 3x^2 - 2x + 1$ at the point where $x = 1$.

    a. **Find Point (x_1, y_1):**
        - Given $x_1 = 1$.
        - $y_1 = 3(1)^2 - 2(1) + 1 = 3 - 2 + 1 = 2$.
        - The point is **(1, 2)**.

    b. **Find Derivative (dy/dx):**
        - $y = 3x^2 - 2x + 1$
        - $dy/dx = 6x - 2$

PAWILLS CONSULTING

c. **Find Tangent Gradient (m_T):**
   - At $x = 1$, $m_T = 6(1) - 2 = 4$.
   - The tangent gradient is **4**.

d. **Equation of Tangent:**
   - $y - y_1 = m_T(x - x_1)$
   - $y - 2 = 4(x - 1)$
   - $y - 2 = 4x - 4$
   - $y = 4x - 2$ (Equation of the Tangent)

e. **Find Normal Gradient (m_N):**
   - $m_N = -1/m_T = -1/4$.

f. **Equation of Normal:**
   - $y - y_1 = m_N(x - x_1)$
   - $y - 2 = -\frac{1}{4}(x - 1)$
   - Multiply by 4: $4(y - 2) = -1(x - 1)$
   - $4y - 8 = -x + 1$
   - $x + 4y - 9 = 0$ (Equation of the Normal)

- **Worked Example 2:** Find the coordinates of the point(s) on the curve $y = x^3 - 12x + 5$ where the tangent is horizontal.

a. **Analyze:** A "horizontal" tangent means its gradient is zero. So, $m_T = 0$.

b. **Find Derivative:**
   - $y = x^3 - 12x + 5$
   - $dy/dx = 3x^2 - 12$

c. **Set derivative to zero:** We are looking for where the gradient $dy/dx = 0$.
   - $3x^2 - 12 = 0$
   - $3x^2 = 12$
   - $x^2 = 4$
   - $x = \pm 2$

d. **Find the y-coordinates:** We have *two* points.

- **At $x = 2$: $y = (2)^3 - 12(2) + 5 = 8 - 24 + 5 = -11$.**
- **At $x = -2$: $y = (-2)^3 - 12(-2) + 5 = -8 + 24 + 5 = 21$.**

- **Answer:** The tangent is horizontal at the points **(2, -11)** and **(-2, 21)**.

EVALUATION

1. What is the mathematical relationship between the derivative (dy/dx) and the tangent line?
2. What is the mathematical relationship between the gradient of the tangent (m_T) and the gradient of the normal (m_N)?
3. Find the gradient of the tangent to the curve $y = 5x^2 - 4x + 7$ at $x = -1$.
4. If the gradient of a tangent at a point is 2/3, what is the gradient of the normal at that point?
5. Find the equation of the tangent to the curve $y = x^2 - 2x$ at the point (3,3).

ASSIGNMENT

1. Find the equation of the **tangent** and the **normal** to the curve $y = x^3 + x^2 - 5$ at the point where $x = 1$.
2. Find the equation of the **normal** to the curve $y = x^2 - 7x + 10$ at the point where the curve *crosses the y-axis*. (Hint: crosses y-axis means $x = 0$).
3. Find the equation of the **tangent** to the curve $y = (x + 2)(x - 1)$ at $x = 2$.
4. Find the coordinates of the point on the curve $y = 4x^2 - 8x + 1$ where the tangent is parallel to the line $y = 4x + 5$. (Hint: "Parallel" means they have the same gradient. Find dy/dx and set it equal to 4).
5. Find the equation of the normal to the curve $y = \frac{4}{x}$ at the point where $x = 2$. (Hint: $y = 4x^{-1}$).

WEEK 3: POLYNOMIALS I

1. DEFINITION OF A POLYNOMIAL

A **polynomial** in a variable x is an algebraic expression that can be written in the general form:

$P(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_1 x + a_0$

Where:

- x is the variable.
- n is a **non-negative integer** (0, 1, 2, 3, ...). This is the most important rule. The exponents *must* be whole numbers.
- $a_n, a_{n-1}, \ldots, a_0$ are the **coefficients** (which are real numbers).
- a_n is the **leading coefficient** (and $a_n \neq 0$).
- a_0 is the **constant term**.

Examples of Polynomials:

- $P(x) = 5x^3 - 2x^2 + 7x + 1$
- $Q(x) = x^4 - 4x$
- $R(x) = 8$ (This is a polynomial of degree 0)

Examples of what are NOT Polynomials:

- $f(x) = 3x^{-2} + 1$ (because the power $n = -2$ is negative)
- $f(x) = \sqrt{x} + 4$ (because $\sqrt{x} = x^{1/2}$, and $n = 1/2$ is not an integer)
- $f(x) = \frac{5}{x}$ (because $5/x = 5x^{-1}$)

---

2. DEGREE OF POLYNOMIALS

The **degree** of a polynomial is simply the **highest power (exponent)** of the variable x that has a non-zero coefficient.

- $P(x) = 7x^4 - 2x + 9 \Rightarrow$ **Degree 4** (a "quartic" polynomial)
- $P(x) = 8x^3 - x^2 \Rightarrow$ **Degree 3** (a "cubic" polynomial)

- $P(x) = x^2 + 5x - 1 \Rightarrow$ **Degree 2** (a "quadratic" polynomial)
- $P(x) = 6x - 7 \Rightarrow$ **Degree 1** (a "linear" polynomial)
- $P(x) = 3 \Rightarrow$ **Degree 0** (a "constant" polynomial)

---

3. OPERATIONS ON POLYNOMIALS

This involves combining polynomials using standard algebra rules. The key is to **combine like terms** (terms with the same power of x).

- **Worked Example 1:** Let $P(x) = 4x^3 - 2x^2 + 5x - 1$ and $Q(x) = 2x^2 + 7x - 3$.
    - **Find $P(x) + Q(x)$:** $(4x^3 - 2x^2 + 5x - 1) + (2x^2 + 7x - 3) = 4x^3 + (-2x^2 + 2x^2) + (5x + 7x) + (-1 - 3) = 4x^3 + 0x^2 + 12x - 4 = \mathbf{4x^3 + 12x - 4}$
    - **Find $P(x) - Q(x)$:** (Be careful with the negative sign) $(4x^3 - 2x^2 + 5x - 1) - (2x^2 + 7x - 3) = 4x^3 - 2x^2 + 5x - 1 - 2x^2 - 7x + 3 = 4x^3 + (-2x^2 - 2x^2) + (5x - 7x) + (-1 + 3) = \mathbf{4x^3 - 4x^2 - 2x + 2}$
- **Worked Example 2:** Let $A(x) = 2x - 3$ and $B(x) = x^2 + 4x - 1$.
    - **Find $A(x) \times B(x)$:** $(2x - 3)(x^2 + 4x - 1) = 2x(x^2 + 4x - 1) - 3(x^2 + 4x - 1) = (2x^3 + 8x^2 - 2x) - (3x^2 + 12x - 3) = 2x^3 + 8x^2 - 2x - 3x^2 - 12x + 3 = 2x^3 + (8x^2 - 3x^2) + (-2x - 12x) + 3 = \mathbf{2x^3 + 5x^2 - 14x + 3}$

---

4. LONG DIVISION OF POLYNOMIALS

This is an essential skill, similar to long division with numbers. It is used to divide a polynomial (the **dividend**) by a smaller polynomial (the **divisor**).

We use this to find the **quotient** and the **remainder**. **Form:** $P(x) = D(x) \times Q(x) + R(x)$ (Dividend = Divisor × Quotient + Remainder)

- **Worked Example 3:** Divide $P(x) = (2x^3 + 5x^2 - 7x + 1)$ by $D(x) = (x - 1)$.

2x^2 + 7x + 0 <-- Quotient (Q(x))

_____

x - 1 | 2x^3 + 5x^2 - 7x + 1 <-- Dividend

-(2x^3 - 2x^2)

_____

0 + 7x^2 - 7x

-(7x^2 - 7x)

_____

0 + 1 <-- Remainder (R(x))

Result: The quotient is $2x^2 + 7x$ and the remainder is **1**. So, $2x^3 + 5x^2 - 7x + 1 = (x - 1)(2x^2 + 7x) + 1$.

- **Worked Example 4 (with a "missing term"):** Divide $(x^3 - 4x + 9)$ by $(x + 2)$.
  - We *must* rewrite the dividend with a $0x^2$ placeholder.
  - $P(x) = x^3 + 0x^2 - 4x + 9$

x^2 - 2x + 0 <-- Quotient (Q(x))

_____

x + 2 | x^3 + 0x^2 - 4x + 9

-(x^3 + 2x^2)

_____

-2x^2 - 4x

-(-2x^2 - 4x)

_____

0 + 9 <-- Remainder (R(x))

Result: The quotient is $x^2 - 2x$ and the remainder is **9**.

EVALUATION

1. Which of the following are *not* polynomials? Explain why.

PAWILLS CONSULTING

- (a) $f(x) = 5x^2 - \frac{2}{x}$
- (b) $f(x) = 7$
- (c) $f(x) = x^3 + 3\sqrt{x}$
- (d) $f(x) = (x - 1)(x + 2)$

2. What is the degree of the polynomial $P(x) = 10x - 3x^4 + 2x^2 - 9$?
3. Given $P(x) = 5x^3 + 2x^2 - 8$ and $Q(x) = 3x^2 + 5x - 4$, find $P(x) - Q(x)$.
4. Perform the long division: $(x^3 - 7x + 6) \div (x + 3)$.
5. When a polynomial $P(x)$ is divided by a divisor $D(x)$, it gives a quotient $Q(x)$ and a remainder $R(x)$. Write the "dividend" equation that links these four terms.

ASSIGNMENT

1. Let $P(x) = x^2 - 3x + 1$ and $Q(x) = 2x^2 - 5$. Find $P(x) \times Q(x)$.
2. Perform the long division: $(3x^4 - 5x^3 + x^2 + 7x - 1) \div (x^2 - x + 1)$. State the quotient and the remainder.
3. A polynomial $P(x)$ of degree 3 has a leading coefficient of 2. A polynomial $Q(x)$ of degree 5 has a leading coefficient of 4.
   - (a) What is the degree of $P(x) + Q(x)$?
   - (b) What is the degree of $P(x) \times Q(x)$?
   - (c) What is the leading coefficient of $P(x) \times Q(x)$?
4. Is the expression $\frac{x^2+1}{x-1}$ a polynomial? Explain your answer.
5. Perform the long division: $(6x^3 + 5x^2 - 13x - 15) \div (2x + 3)$.

WEEK 4: POLYNOMIALS II

1. THE REMAINDER THEOREM

Introduction: In Week 3, we used long division to find the remainder. This is a very long and slow process. The **Remainder Theorem** provides a fast, elegant "shortcut" to find the remainder *without* doing the division.

The Theorem: If a polynomial $P(x)$ is divided by a linear divisor $(x - a)$, the remainder R is equal to $P(a)$.

Proof (The "Why"):

1. From the long division rule, we know: $P(x) = D(x) \times Q(x) + R(x)$
2. In this case, our divisor $D(x)$ is the linear term $(x - a)$. The remainder R will be a constant. $P(x) = (x - a) \times Q(x) + R$
3. This equation must be true for *all* values of x. Let's test the specific value $x = a$.
4. Substitute $x = a$ into the equation: $P(a) = (a - a) \times Q(a) + R$ $P(a) = (0) \times Q(a) + R$ $P(a) = 0 + R$ $P(a) = R$
5. This proves that to find the remainder, you just need to "plug in" the root of the divisor.

- **Worked Example 1:** Find the remainder when $P(x) = x^3 + 2x^2 - 5x - 6$ is divided by $(x - 2)$.
    - **Divisor:** $x - 2$. To find a, set the divisor to zero: $x - 2 = 0 \Rightarrow x = 2$. So, $a = 2$.
    - **Remainder** = $P(a) = P(2)$**:** $P(2) = (2)^3 + 2(2)^2 - 5(2) - 6$ $P(2) = 8 + 2(4) - 10 - 6$ $P(2) = 8 + 8 - 10 - 6 = 16 - 16 = 0$
    - **Answer:** The remainder is **0**.
- **Worked Example 2:** Find the remainder when $P(x) = 2x^3 - x^2 + 4x - 1$ is divided by $(x + 1)$.
    - **Divisor:** $x + 1$. Set to zero: $x + 1 = 0 \Rightarrow x = -1$. So, $a = -1$.

- **Remainder** = $P(a) = P(-1)$: $P(-1) = 2(-1)^3 - (-1)^2 + 4(-1) - 1$ $P(-1) = 2(-1) - (1) - 4 - 1$ $P(-1) = -2 - 1 - 4 - 1 = -8$
- **Answer:** The remainder is **-8**.

- **Worked Example 3 (Fractional Root):** Find the remainder when $P(x) = 3x^3 - 5x + 1$ is divided by $(2x - 1)$.
  - **Divisor:** $2x - 1$. Set to zero: $2x - 1 = 0 \Rightarrow 2x = 1 \Rightarrow x = 1/2$. So, $a = 1/2$.
  - **Remainder** = $P(a) = P(1/2)$: $P(1/2) = 3(\frac{1}{2})^3 - 5(\frac{1}{2}) + 1$ $P(1/2) = 3(\frac{1}{8}) - \frac{5}{2} + 1$ $P(1/2) = \frac{3}{8} - \frac{20}{8} + \frac{8}{8} = \frac{3-20+8}{8} = -\frac{9}{8}$
  - **Answer:** The remainder is **-9/8**.

---

2. THE FACTOR THEOREM

Introduction: The **Factor Theorem** is a direct and powerful consequence of the Remainder Theorem.

- If a number (like 12) is divided by (4), the remainder is 0. This means 4 is a **factor** of 12.
- The same is true for polynomials. If a polynomial $P(x)$ is divided by $(x - a)$ and the remainder R is 0, then $(x - a)$ must be a **factor** of $P(x)$.

The Theorem: $(x - a)$ is a **factor** of a polynomial $P(x)$ *if and only if* $P(a) = 0$.

- ($P(a) = 0$ means a is a **root** of the equation $P(x) = 0$).
- **Worked Example 4:** Show that $(x - 1)$ is a factor of $P(x) = x^3 + 2x^2 - x - 2$.
  - **Divisor:** $x - 1$. The root is $x = 1$. So, $a = 1$.
  - **Test** $P(1)$: $P(1) = (1)^3 + 2(1)^2 - (1) - 2$ $P(1) = 1 + 2 - 1 - 2 = 3 - 3 = 0$
  - **Conclusion:** Since $P(1) = 0$, the remainder is 0. Therefore, $(x - 1)$ **is a factor** of $P(x)$.

---

3. APPLICATIONS IN SIMPLIFYING (FINDING COEFFICIENTS)

The theorems are most powerful when used to find *unknown coefficients*.

- **Worked Example 5:** The polynomial $P(x) = x^3 + kx^2 - 4x - 12$ has $(x + 2)$ as a factor. Find the value of k.

    a. **Analyze:** The Factor Theorem states that if $(x + 2)$ is a factor, then $P(-2)$ must be equal to 0.

    b. **Set up the equation:**

        - $P(-2) = 0$
        - $(-2)^3 + k(-2)^2 - 4(-2) - 12 = 0$

    c. **Solve for k:**

        - $-8 + k(4) + 8 - 12 = 0$
        - $-8 + 4k + 8 - 12 = 0$
        - $4k - 12 = 0$
        - $4k = 12$
        - $k = 3$

    - **Answer:** The value of k is **3**.

- **Worked Example 6 (Two Unknowns):** Let $P(x) = 2x^3 + ax^2 + bx + 1$. When $P(x)$ is divided by $(x - 2)$, the remainder is 27. When divided by $(x + 1)$, the remainder is 0. Find a and b.

    a. **Analyze:** We have two unknowns, so we need two equations.

        - R. Theorem: $P(2) = 27$
        - F. Theorem: $P(-1) = 0$

    b. **Form Equation 1 ($P(2) = 27$):**

        - $2(2)^3 + a(2)^2 + b(2) + 1 = 27$
        - $2(8) + 4a + 2b + 1 = 27$
        - $16 + 4a + 2b + 1 = 27$
        - $4a + 2b = 27 - 17$

- $4a + 2b = 10 \Rightarrow \mathbf{2a + b = 5}$ ... (Eq 1)

c. **Form Equation 2 ($P(-1) = 0$):**

- $2(-1)^3 + a(-1)^2 + b(-1) + 1 = 0$
- $2(-1) + a(1) - b + 1 = 0$
- $-2 + a - b + 1 = 0$
- $a - b - 1 = 0 \Rightarrow \mathbf{a - b = 1}$ ... (Eq 2)

d. **Solve the simultaneous equations:**

- $2a + b = 5$
- $a - b = 1$
- Add (Eq 1) + (Eq 2):
- $3a = 6 \Rightarrow \mathbf{a = 2}$
- Substitute $a = 2$ into (Eq 2):
- $2 - b = 1 \Rightarrow b = 1$

- **Answer:** $a = 2$ and $b = 1$.

EVALUATION

1. What is the Remainder Theorem?
2. What is the Factor Theorem?
3. Without performing long division, find the remainder when $x^3 - 3x^2 + 5x + 2$ is divided by $(x - 1)$.
4. Show that $(x + 3)$ is a factor of $x^3 + 3x^2 - 4x - 12$.
5. If $P(x) = x^3 + ax - 10$ and the remainder when $P(x)$ is divided by $(x - 2)$ is 4, find the value of a.

ASSIGNMENT

1. When the polynomial $f(x) = 2x^3 + x^2 + kx - 15$ is divided by $(x - 2)$, the remainder is 21. Find the value of k.
2. Given $P(x) = 6x^3 + 7x^2 + ax + b$. If $(x - 1)$ is a factor, and $P(x)$ leaves a remainder of 10 when divided by $(x + 1)$, find the values of a and b.

3. What is the remainder when $4x^3 - 2x^2 + 5x - 1$ is divided by $(2x - 1)$?
4. Use the Factor Theorem to show that $(x - 1)$, $(x - 2)$, and $(x - 3)$ are all factors of $P(x) = x^3 - 6x^2 + 11x - 6$.
5. Explain the relationship between the Remainder Theorem, the Factor Theorem, and the "roots" of a polynomial equation.

PAWILLS CONSULTING

WEEK 6: POLYNOMIALS II (CONTINUED)

1. APPLICATION OF FACTOR AND REMAINDER THEOREM

This week, we synthesize the skills from Weeks 3 and 4 to solve the primary goal of this topic: **solving higher-order equations**.

2. SOLVING HIGHER ORDER EQUATIONS (E.G., CUBIC EQUATIONS)

A "cubic equation" is a polynomial equation of degree 3: $ax^3 + bx^2 + cx + d = 0$. A cubic equation will *always* have **3 roots** (solutions). These can be:

- Three real roots.
- One real root and two complex (imaginary) roots.

Our goal is to find the real roots. We do this by *reducing* the cubic (degree 3) problem into a quadratic (degree 2) problem, which we already know how to solve (from Week 1).

The 3-Step Strategy:

1. **FIND ONE ROOT (using the Factor Theorem):** Use "trial and error" to find one simple, integer root. You only need to test factors of the constant term.
2. **REDUCE (using Long Division):** Divide the cubic polynomial by the linear factor you just found. The result will be a quadratic quotient (and the remainder must be 0).
3. **SOLVE (using Quadratic Methods):** Solve the remaining quadratic equation (by factorization or formula) to find the other two roots.

WORKED EXAMPLES

- **Worked Example 1 (Simple Roots):** Find all the roots of the equation $P(x) = x^3 + 2x^2 - 5x - 6 = 0$.

    a. **Find one root (Factor Theorem):**

        - Constant term is -6. Possible roots: $\pm 1, \pm 2, \pm 3, \pm 6$.
        - Test $x = 1$: $P(1) = 1 + 2 - 5 - 6 = -8 \neq 0$.

- Test $x = -1$: $P(-1) = (-1)^3 + 2(-1)^2 - 5(-1) - 6 = -1 + 2 + 5 - 6 = 0$.
- **Conclusion:** Since $P(-1) = 0$, we know that $x = -1$ is a root, and $(x + 1)$ **is a factor**.

b. **Reduce (Long Division):** Divide $P(x)$ by $(x + 1)$.

```
      x^2  + x  - 6
     ____________________
x + 1 | x^3 + 2x^2 - 5x - 6
      -(x^3 + x^2)
     ____________________
          0 + x^2 - 5x
          -(x^2 +  x)
          ___________
              -6x - 6
              -(-6x - 6)
              __________
                    0
```

c. **Solve the quadratic:**

- The quotient is $x^2 + x - 6 = 0$.
- We can factorize this: two numbers that multiply to -6 and add to 1 are +3 and -2.
- $(x + 3)(x - 2) = 0$
- The roots are $x = -3$ and $x = 2$.

o **Final Answer:** The three real roots are $x = -1, x = 2, x = -3$.

- **Worked Example 2 (Using Quadratic Formula):** Solve the equation $P(x) = x^3 - 5x^2 + 8x - 6 = 0$.

a. **Find one root (Factor Theorem):**

- Constant term is -6. Possible roots: $\pm 1, \pm 2, \pm 3, \pm 6$.
- Test $x = 1$: $P(1) = 1 - 5 + 8 - 6 = -2 \neq 0$.

- Test $x = 2$: $P(2) = 8 - 5(4) + 8(2) - 6 = 8 - 20 + 16 - 6 = -2 \neq 0$.
- Test $x = 3$: $P(3) = (3)^3 - 5(3)^2 + 8(3) - 6 = 27 - 5(9) + 24 - 6 = 27 - 45 + 24 - 6 = 0$.
- **Conclusion:** Since $P(3) = 0$, we know that $x = 3$ is a root, and $(x - 3)$ **is a factor**.

b. **Reduce (Long Division):** Divide $P(x)$ by $(x - 3)$.

```
        x^2  - 2x  + 2
      ____________________
x - 3 | x^3 - 5x^2 + 8x - 6
        -(x^3 - 3x^2)
        ____________________
             -2x^2 + 8x
             -(-2x^2 + 6x)
             ____________________
                  2x - 6
                  -(2x - 6)
                  _________
                        0
```

c. **Solve the quadratic:**

- The quotient is $x^2 - 2x + 2 = 0$.
- Let's test the discriminant ($b^2 - 4ac$): $(-2)^2 - 4(1)(2) = 4 - 8 = -4$.
- Since the discriminant is negative, this quadratic **has no real roots**. The roots are complex.

- **Final Answer:** The *only* real root of this equation is $x = 3$.

EVALUATION

1. What is the general 3-step strategy for solving a cubic equation?
2. If you are solving $P(x) = 2x^3 - x^2 - 8x + 4 = 0$, what are the possible integer roots you should test using the Factor Theorem?

3. Given that $(x-1)$ is a factor of $P(x) = x^3 - 4x^2 + x + 2$, find the other two factors.
4. Solve the equation $x^3 + 2x^2 - 5x - 6 = 0$ completely (Note: this is the same as Example 1, use a different root to start, e.g., $x = 2$).
5. Find the roots of $x^3 - 7x - 6 = 0$.

ASSIGNMENT

1. Find all three real roots of the equation $P(x) = x^3 + 4x^2 + x - 6 = 0$.
2. Find all three real roots of the equation $P(x) = 2x^3 + 3x^2 - 8x + 3 = 0$. (Hint: $x = 1$ is a root).
3. Given that $f(x) = x^3 + ax^2 + bx - 6$. If $(x-1)$ is a factor, and $f(x)$ leaves a remainder of -12 when divided by $(x+1)$, find the values of a and b, and then solve $f(x) = 0$ completely.
4. One root of the equation $x^3 + 6x^2 - x - 30 = 0$ is $x = 2$. Find the other two roots.
5. Find the *real* root(s) of the equation $x^3 + x - 10 = 0$.

PAWILLS CONSULTING

WEEK 7: MIDTERM BREAK

This week is for the Midterm Break. There are no formal lesson notes.

PAWILLS CONSULTING

WEEK 8: CUBIC EQUATIONS AND THEIR FACTORIZATION

1. REVIEW OF SOLVING CUBIC EQUATIONS

As covered in Week 6, the primary method for solving a general cubic equation $ax^3 + bx^2 + cx + d = 0$ is:

1. Use the **Factor Theorem** to find one root, a. This means $(x - a)$ is a factor.
2. Use **Long Division** to divide the cubic by $(x - a)$, resulting in a quadratic quotient.
3. Solve the remaining **quadratic equation** to find the other two roots.

---

2. FACTORIZATION OF SPECIAL CUBIC EXPRESSIONS

Some cubic expressions follow a specific pattern and can be factorized directly, without needing the Factor Theorem.

A. DIFFERENCE OF TWO CUBES

- **Formula:** $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$
- **Mnemonic (SOAP):** This helps remember the signs.
  - $(a$ [Same Sign]$b)(a^2$ [Opposite Sign]ab [Always Positive]$b^2)$
- **Worked Example 1:** Factorize $x^3 - 8$.
  - This is $x^3 - 2^3$. Here, $a = x$ and $b = 2$.
  - Using the formula: $(x - 2)(x^2 + (x)(2) + 2^2)$
  - $= \mathbf{(x - 2)(x^2 + 2x + 4)}$
- **Worked Example 2:** Factorize $27y^3 - 64w^3$.
  - This is $(3y)^3 - (4w)^3$. Here, $a = 3y$ and $b = 4w$.
  - Using the formula: $(3y - 4w)((3y)^2 + (3y)(4w) + (4w)^2)$
  - $= \mathbf{(3y - 4w)(9y^2 + 12yw + 16w^2)}$

B. SUM OF TWO CUBES

- **Formula:** $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$

- **Mnemonic (SOAP):**
    - (a [Same Sign]b)(a^2 [Opposite Sign]ab [Always Positive]b^2)
- **Worked Example 3:** Factorize $x^3 + 125$.
    - This is $x^3 + 5^3$. Here, $a = x$ and $b = 5$.
    - Using the formula: $(x + 5)(x^2 - (x)(5) + 5^2)$
    - $= \mathbf{(x + 5)(x^2 - 5x + 25)}$
- **Worked Example 4:** Factorize $8x^3 + 1$.
    - This is $(2x)^3 + 1^3$. Here, $a = 2x$ and $b = 1$.
    - Using the formula: $(2x + 1)((2x)^2 - (2x)(1) + 1^2)$
    - $= \mathbf{(2x + 1)(4x^2 - 2x + 1)}$

---

3. RELATIONSHIP BETWEEN ROOTS AND COEFFICIENTS (VIETA'S FORMULAS)

For a general cubic equation $ax^3 + bx^2 + cx + d = 0$, let the three roots be α **(alpha),** β **(beta), and** γ **(gamma)**.

We know the factored form must be $a(x - \alpha)(x - \beta)(x - \gamma) = 0$. By expanding the factored form and comparing its coefficients to the general form $x^3 + (b/a)x^2 + (c/a)x + (d/a) = 0$, we can derive a set of relationships.

The Relationships (Vieta's Formulas for Cubics):

1. **Sum of roots (one at a time):** $\alpha + \beta + \gamma = -\frac{b}{a}$
2. **Sum of roots (taken two at a time):** $\alpha\beta + \beta\gamma + \alpha\gamma = \frac{c}{a}$
3. **Product of roots:** $\alpha\beta\gamma = -\frac{d}{a}$

- **Worked Example 5:** Given the equation $2x^3 - 4x^2 - 10x + 8 = 0$, find the sum of its roots, the sum of the product of its roots taken two at a time, and the product of its roots.
    a. **Identify coefficients:** $a = 2, b = -4, c = -10, d = 8$
    b. **Sum of roots ($\alpha + \beta + \gamma$):** $-\frac{b}{a} = -\frac{(-4)}{2} = \frac{4}{2} = \mathbf{2}$

c. **Sum of product of roots ($\alpha\beta + \beta\gamma + \alpha\gamma$):** $\frac{c}{a} = \frac{-10}{2} = -\mathbf{5}$

d. **Product of roots ($\alpha\beta\gamma$):** $-\frac{d}{a} = -\frac{8}{2} = -\mathbf{4}$

- **Worked Example 6: Forming an equation given its roots Problem:** Form the cubic equation whose roots are 1, 2, and 3.

  - **Method 1 (Factorization):**
    - The factors are $(x-1), (x-2), (x-3)$.
    - $(x-1)(x-2) = x^2 - 3x + 2$
    - $(x^2 - 3x + 2)(x-3) = x(x^2 - 3x + 2) - 3(x^2 - 3x + 2)$
    - $= (x^3 - 3x^2 + 2x) - (3x^2 - 9x + 6)$
    - $= x^3 - 6x^2 + 11x - 6 = 0$
  - **Method 2 (Vieta's Formulas):**
    - The equation is $x^3 - (\text{sum})x^2 + (\text{sum of products})x - (\text{product}) = 0$
    - $\alpha = 1, \beta = 2, \gamma = 3$
    - Sum: $1 + 2 + 3 = 6$
    - Sum of products: $(1)(2) + (2)(3) + (1)(3) = 2 + 6 + 3 = 11$
    - Product: $(1)(2)(3) = 6$
    - Equation: $x^3 - (6)x^2 + (11)x - (6) = 0$
    - $\Rightarrow \mathbf{x^3 - 6x^2 + 11x - 6 = 0}$

EVALUATION

1. Factorize the expression $y^3 - 64$.
2. Factorize the expression $8x^3 + 1$.
3. Given the equation $x^3 - 4x^2 - 7x + 10 = 0$. Find:
   - (a) The sum of the roots ($\alpha + \beta + \gamma$).
   - (b) The product of the roots ($\alpha\beta\gamma$).
4. If the roots of $2x^3 + 3x^2 - 11x - 6 = 0$ are α, β, γ, find the value of $\alpha\beta + \beta\gamma + \alpha\gamma$.
5. Form the cubic equation whose roots are $-1, 1, 4$.

ASSIGNMENT

1. Given that two of the roots of $x^3 - 2x^2 - 5x + 6 = 0$ are 1 and 3. Find the third root *without* solving the equation. (Hint: Use sum or product of roots).
2. Factorize $x^6 - 64$ completely. (Hint: Treat it as a "difference of two squares" first, $(x^3)^2 - 8^2$, then as a "difference/sum of two cubes").
3. The roots of the equation $x^3 + px^2 + qx + 10 = 0$ are α, 2α, and 5. Find the values of α, p, and q. (Hint: Start with the product of roots).
4. Solve the equation $x^3 - 1 = 0$. (Hint: Factorize it and use the quadratic formula on the non-linear factor to find the complex roots).
5. If the roots of $x^3 + 2x^2 - 3x - 1 = 0$ are α, β, γ, find the value of $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$. (Hint: Find a common denominator).

PAWILLS CONSULTING

WEEK 9: LOGICAL REASONING I

1. STATEMENTS

- **Definition:** A **statement** (or proposition) is a declarative sentence that is either **True (T)** or **False (F)**, but *not both* simultaneously. This T or F value is called its **Truth Value**.
- **Notation:** We use lowercase letters like p, q, r to represent statements.
- **Examples of Statements:**
  - p: "Lagos is in Nigeria." (Truth Value = T)
  - q: "$3 \times 5 = 18$." (Truth Value = F)
  - r: "The capital of Nigeria is Abuja." (Truth Value = T)
  - s: "All quadratic equations have two real roots." (Truth Value = F, because of the discriminant).
- **Examples of NON-Statements (Sentences that are not propositions):**
  - "How are you?" (This is a question)
  - "Run!" (This is a command)
  - "$x + 2 = 5$" (This is an "open" sentence; its truth value depends on x)
  - "This statement is false." (This is a paradox)

---

2. TRUTH TABLES

A **truth table** is a diagram that shows the resulting truth value of a compound statement for all possible combinations of truth values of its simple components.

- If there is 1 statement (p), there are $2^1 = 2$ rows.
- If there are 2 statements (p, q), there are $2^2 = 4$ rows.
- If there are 3 statements (p, q, r), there are $2^3 = 8$ rows.
- The standard setup for 4 rows is:

p	q

p	q
T	T
T	F
F	T
F	F

---

3. LOGICAL OPERATIONS (CONNECTIVES)

A. NEGATION (Symbol: ~ or ¬)

- **Meaning:** "not". The negation of a statement p (written $\sim p$) is a new statement that has the *opposite* truth value of p.
- **Worked Example 1:**
  - p: "It is raining." (T)
  - $\sim p$: "It is not raining." (F)
- **Truth Table for Negation:**

p	$\sim p$
T	F
F	T

B. CONJUNCTION (Symbol: ∧)

- **Meaning:** "and". A compound statement $p \wedge q$ ("p and q") is only true if *both* p and q are true.
- **Worked Example 2:** "Lagos is in Nigeria (T) **and** $2 + 2 = 4$ (T)." ⇒ The whole statement is **T**.
- **Worked Example 3:** "Lagos is in Nigeria (T) **and** $2 + 2 = 5$ (F)." ⇒ The whole statement is **F**.
- **Truth Table for Conjunction:**

p	q	$p \wedge q$

PAWILLS CONSULTING

p	q	$p \wedge q$
T	T	**T**
T	F	**F**
F	T	**F**
F	F	**F**

C. DISJUNCTION (Symbol: ∨)

- **Meaning:** "or". This is the **inclusive or**. A compound statement $p \vee q$ ("p or q") is true if *at least one* of p or q is true. It is only false if *both* are false.
- **Worked Example 4:** "The student is male (T) **or** the student is 16 (F)." ⇒ The whole statement is **T**.
- **Worked Example 5:** "The student is female (F) **or** the student is 12 (F)." ⇒ The whole statement is **F**.
- **Truth Table for Disjunction:**

p	q	$p \vee q$
T	T	**T**
T	F	**T**
F	T	**T**
F	F	**F**

---

4. CONSTRUCTING A TRUTH TABLE

- **Worked Example 6:** Construct a truth table for the compound statement $(p \wedge q) \vee (\sim p)$.
    a. We have 2 variables (p, q), so we need $2^2 = 4$ rows.
    b. We need columns for our "building blocks": p, q, $\sim p$, $(p \wedge q)$.
    c. We need a final column for the full statement: $(p \wedge q) \vee (\sim p)$.

p	q	$\sim p$	$(p \wedge q)$	$(p \wedge q) \vee (\sim p)$
T	T	F	T ∧ T = T	T ∨ F = **T**
T	F	F	T ∧ F = F	F ∨ F = **F**
F	T	T	F ∧ T = F	F ∨ T = **T**
F	F	T	F ∧ F = F	F ∨ T = **T**

EVALUATION

1. Which of the following are valid logical statements?
    - (a) $2 + 7 = 9$
    - (b) Close the door.
    - (c) $x > 5$
    - (d) All triangles have three sides.
2. Let p be "The sun is hot" and q be "The moon is made of cheese." Write the symbolic form for:
    - (a) "The sun is hot and the moon is made of cheese."
    - (b) "The sun is not hot or the moon is made of cheese."
    - (c) "The sun is hot and the moon is not made of cheese."
3. If p is True and q is False, what is the truth value of $(p \vee q) \wedge (\sim p)$?
4. Construct the truth table for the statement $p \wedge \sim q$.
5. What is the "inclusive or" (disjunction)? When is $p \vee q$ false?

ASSIGNMENT

1. Let p be "Bola is tall" and q be "Bola is smart." Write the following statements in symbolic form:
    - (a) Bola is tall and smart.
    - (b) Bola is tall but not smart.
    - (c) Bola is neither tall nor smart. (Hint: $\sim p \wedge \sim q$)
    - (d) It is not true that Bola is tall and smart. (Hint: $\sim (p \wedge q)$)

2. Construct a truth table for the compound statement $(p \vee q) \wedge \sim p$.
3. Construct a truth table for the compound statement $\sim (p \wedge q) \vee (\sim p \vee q)$.
4. If p is True, q is False, and r is True, find the truth value of $(\sim p \vee q) \wedge (r \vee q)$.
5. How many rows would a truth table for the statement $(p \vee q) \wedge (r \vee s)$ require?

WEEK 10: LOGICAL REASONING II

1. CONDITIONAL STATEMENT (Symbol: → or ⇒)

- **Meaning:** "if... then...". A conditional statement $p \rightarrow q$ ("if p, then q") links an **antecedent** (or hypothesis) p with a **consequent** (or conclusion) q.
- **Example:** "If it is raining (p), then the ground is wet (q)."
- **Key Property (The Only False Case):** A conditional statement is *only* considered **False** when a **True** antecedent leads to a **False** consequent.
    - If I promise "If you get an A (p), I will give you N1000 (q)", the *only* way I have lied is if you *get an A (T)* and I *do not give you N1000 (F)*.
    - If you *don't get an A (F)*, my promise is not broken, regardless of whether I give you N1000 (T) or not (F).
- **Truth Table for Conditional:**

p	q	$p \rightarrow q$
T	T	**T**
T	F	**F**
F	T	**T**
F	F	**T**

2. BICONDITIONAL STATEMENT (Symbol: ↔ or ⇔)

- **Meaning:** "...if and only if..." (often written "iff"). A biconditional statement $p \leftrightarrow q$ means $p \rightarrow q$ *and* $q \rightarrow p$.
- **Example:** "A triangle is equilateral (p) **if and only if** it is equiangular (q)."
- **Key Property:** The statement is only **True** if both p and q have the **same truth value** (both True or both False).
- **Truth Table for Biconditional:**

PAWILLS CONSULTING

p	q	$p \leftrightarrow q$
T	T	**T**
T	F	**F**
F	T	**F**
F	F	**T**

---

3. TAUTOLOGY AND CONTRADICTION

- **TAUTOLOGY:**
  - **Definition:** A tautology is a compound statement that is **ALWAYS TRUE**, regardless of the truth values of its simple components. The final column of its truth table is all "T".
  - **Worked Example 1:** Prove $p \vee (\sim p)$ is a tautology.

p	$\sim p$	$p \vee (\sim p)$
T	F	T ∨ F = **T**
F	T	F ∨ T = **T**

  - Since the final column is all "T", the statement is a tautology.
- **CONTRADICTION (OR FALLACY):**
  - **Definition:** A contradiction is a compound statement that is **ALWAYS FALSE**, regardless of the truth values of its simple components. The final column of its truth table is all "F".
  - **Worked Example 2:** Prove $p \wedge (\sim p)$ is a contradiction.

p	$\sim p$	$p \wedge (\sim p)$
T	F	T ∧ F = **F**
F	T	F ∧ T = **F**

- Since the final column is all "F", the statement is a contradiction.

---

4. INVERSE, CONVERSE, AND CONTRAPOSITIVE

For any conditional statement $p \rightarrow q$, we can form three related statements.

- **Worked Example 3:**
  - **Original Statement ($p \rightarrow q$):** "If it is raining, then the ground is wet."
  - **Converse ($q \rightarrow p$):** "If the ground is wet, then it is raining." (This is not necessarily true - I could be watering the ground).
  - **Inverse ($\sim p \rightarrow \sim q$):** "If it is not raining, then the ground is not wet." (Also not necessarily true).
  - **Contrapositive ($\sim q \rightarrow \sim p$):** "If the ground is not wet, then it is not raining." (This *is* true).
- **Key Property (Logical Equivalence):**
  - The **Original Statement ($p \rightarrow q$)** and its **Contrapositive ($\sim q \rightarrow \sim p$)** are **logically equivalent** (they always have the same truth value).
  - The **Converse ($q \rightarrow p$)** and the **Inverse ($\sim p \rightarrow \sim q$)** are also logically equivalent to each other.

EVALUATION

1. A conditional statement $p \rightarrow q$ is only false in one specific case. What is it?
2. A biconditional statement $p \leftrightarrow q$ is only true in two specific cases. What are they?
3. What is a tautology? Give a simple example.
4. What is a contradiction? Give a simple example.
5. What is the **contrapositive** of the statement "If you study hard, you will pass"?

ASSIGNMENT

1. Construct a full truth table for the statement $(p \to q) \wedge (q \to p)$. What other connective has an identical truth table? (This proves $p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$).
2. Construct a full truth table to prove that $p \to q$ is logically equivalent to its contrapositive $\sim q \to \sim p$.
3. If p is False and q is True, what is the truth value of $(p \leftrightarrow q) \to (\sim p \wedge q)$?
4. Write the converse, inverse, and contrapositive of the statement: "If $x = 2$, then $x^2 = 4$." Which of these are true?
5. What is the "exclusive or" (XOR)? (Research this). How does its truth table differ from the disjunction $(p \vee q)$?

PAWILLS CONSULTING

WEEK 11: TRIGONOMETRIC FUNCTION

1. DEFINITION (SOH CAH TOA)

Trigonometry is the study of the relationship between the sides and angles of a **right-angled triangle**.

For a given angle θ (theta) in a right-angled triangle:

- **Opposite (Opp):** The side directly across from the angle θ.
- **Adjacent (Adj):** The side next to the angle θ (that is *not* the hypotenuse).
- **Hypotenuse (Hyp):** The longest side, opposite the $90°$ angle.

The Three Primary Trigonometric Ratios:

- **Sine (sin):** $\sin(\theta) = \frac{\text{Opposite}}{\text{Hypotenuse}}$ (SOH)
- **Cosine (cos):** $\cos(\theta) = \frac{\text{Adjacent}}{\text{Hypotenuse}}$ (CAH)
- **Tangent (tan):** $\tan(\theta) = \frac{\text{Opposite}}{\text{Adjacent}}$ (TOA)

The Three Reciprocal Ratios:

- **Cosecant (cosec or csc):** $\csc(\theta) = \frac{1}{\sin(\theta)} = \frac{\text{Hypotenuse}}{\text{Opposite}}$
- **Secant (sec):** $\sec(\theta) = \frac{1}{\cos(\theta)} = \frac{\text{Hypotenuse}}{\text{Adjacent}}$
- **Cotangent (cot):** $\cot(\theta) = \frac{1}{\tan(\theta)} = \frac{\text{Adjacent}}{\text{Opposite}}$

2. VALUES OF TRIGONOMETRIC RATIOS

2A. SPECIAL ANGLES ($30°, 45°, 60°$)

We can find the *exact* values (not decimals) for these angles using two special triangles.

1. **The $45° - 45° - 90°$ Triangle:**

- An isosceles right-angled triangle with two sides of length 1.
- By Pythagoras, Hypotenuse = $\sqrt{1^2 + 1^2} = \sqrt{2}$.
- $\sin(45°) = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$
- $\cos(45°) = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$
- $\tan(45°) = \frac{1}{1} = 1$

2. **The** $30° - 60° - 90°$ **Triangle:**
   - Start with an equilateral triangle of side length 2. Cut it in half.
   - The angles are 30°, 60°, 90°.
   - The sides are: Hyp = 2, Opp to 30° = 1, Adj to 30° = $\sqrt{2^2 - 1^2} = \sqrt{3}$.
   - **For** $30°$**:** $\sin(30°) = 1/2$, $\cos(30°) = \sqrt{3}/2$, $\tan(30°) = 1/\sqrt{3}$
   - **For** $60°$**:** $\sin(60°) = \sqrt{3}/2$, $\cos(60°) = 1/2$, $\tan(60°) = \sqrt{3}/1 = \sqrt{3}$

- **Worked Example 1:** Find the exact value of $\frac{\sin(60°)+\cos(30°)}{\tan(45°)}$.
  - $\sin(60°) = \sqrt{3}/2$
  - $\cos(30°) = \sqrt{3}/2$
  - $\tan(45°) = 1$
  - Value = $\frac{(\sqrt{3}/2+\sqrt{3}/2)}{1} = \frac{(2\sqrt{3}/2)}{1} = \sqrt{3}$.

2B. QUADRANTAL ANGLES & THE ASTC (CAST) RULE

We use the **Unit Circle** (a circle with radius $r = 1$, center (0,0)).

- A point (x, y) on the circle is defined by $x = \cos(\theta)$ and $y = \sin(\theta)$.
- This gives us the signs in the four quadrants:
  - **Quadrant I (**$0° - 90°$**):** (x, y) = (+, +) ⇒ **A**ll ratios are positive.
  - **Quadrant II (**$90° - 180°$**):** (x, y) = (-, +) ⇒ Only **S**ine is positive.
  - **Quadrant III (**$180° - 270°$**):** (x, y) = (-, -) ⇒ Only **T**angent is positive.
  - **Quadrant IV (**$270° - 360°$**):** (x, y) = (+, -) ⇒ Only **C**osine is positive.

- **Worked Example 2:** Find the exact value of $\cos(225°)$.
    a. **Quadrant:** $225°$ is in Quadrant III.
    b. **Sign (ASTC):** In Q.III, only Tangent is positive, so $\cos(225°)$ is **negative**.
    c. **Related Acute Angle:** $\theta_R = 225° - 180° = 45°$.
    d. **Solve:** $\cos(225°) = -\cos(45°) = -\frac{1}{\sqrt{2}}$ or $-\frac{\sqrt{2}}{2}$.

---

3. ANGLE OF ELEVATION AND DEPRESSION

These are applications of SOH CAH TOA in word problems.

- **Angle of Elevation:** The angle an observer *looks up* from the horizontal to see an object.
- **Angle of Depression:** The angle an observer *looks down* from the horizontal to see an object.

[Image of angle of elevation and depression diagram]

- **Key Property:** The Angle of Elevation from (A to B) is *equal* to the Angle of Depression from (B to A), because they are **alternate interior angles**.
- **Worked Example 3:** From the top of a cliff 100m high, the angle of depression to a boat at sea is $30°$. How far is the boat from the base of the cliff?
    a. **Draw:** Draw a right-angled triangle.
        - The cliff height (Opposite to the angle at the boat) = 100m.
        - The distance to the boat (Adjacent to the angle at the boat) = x.
        - The angle of depression from the top is $30°$.
    b. **Find the inner angle:** The alternate interior angle *at the boat* is also $30°$.
    c. **Set up the ratio:** We have Opposite (100) and Adjacent (x), so we use TOA.
        - $\tan(30°) = \frac{\text{Opposite}}{\text{Adjacent}}$
        - $\tan(30°) = \frac{100}{x}$
    d. **Solve for x:**

        - $x = \frac{100}{\tan(30°)}$
        - We know $\tan(30°) = \frac{1}{\sqrt{3}}$
        - $x = \frac{100}{(1/\sqrt{3})} = 100\sqrt{3}$ meters.
    - **Answer:** $x \approx 173.2$ meters.
- **Worked Example 4:** A man stands 50m away from the base of a flagpole. He measures the angle of elevation to the top of the pole as $40°$. Find the height of the flagpole.
    a. **Draw:**
        - Height of pole (Opposite) = h.
        - Distance from base (Adjacent) = 50m.
        - Angle = $40°$.
    b. **Set up ratio (TOA):**
        - $\tan(40°) = \frac{h}{50}$
    c. **Solve for h:**
        - $h = 50 \times \tan(40°)$
        - $h = 50 \times 0.8391$
    - **Answer:** $h \approx 41.95$ meters.

EVALUATION

1. What are the six trigonometric ratios, and which three are the "reciprocal" ratios?
2. Find the *exact* value (no calculators) of $\sin(60°)$, $\cos(45°)$, and $\tan(30°)$.
3. Using the ASTC rule, find the value of $\sin(300°)$.
4. If $\cos(\theta) = 3/5$ and θ is in Quadrant IV, find $\sin(\theta)$ and $\tan(\theta)$.
5. What is the difference between an angle of elevation and an angle of depression?

ASSIGNMENT

1. A ladder 10m long leans against a vertical wall. It makes an angle of $60°$ with the ground.
    - (a) How high up the wall does the ladder reach?

- (b) How far is the base of the ladder from the wall?

2. From a window 15m above the ground, the angle of depression to the base of a tree is 25° and the angle of elevation to the top of the tree is 35°.
   - (a) Find the horizontal distance from the window to the tree.
   - (b) Find the total height of the tree.
3. Find all angles θ between 0° and 360° that satisfy the equation $\sin(\theta) = -1/2$.
4. Prove the identity: $\sec(\theta)/\tan(\theta) = \csc(\theta)$.
5. A boat sails 10km East, then 12km North. What is its final bearing from the starting point? (e.g., N x° E).

PAWILLS CONSULTING

WEEK 12: GRAPHS OF TRIGONOMETRIC FUNCTION

1. THE SINE FUNCTION: $y = \sin(x)$

- **Properties:**
  - **Period:** The graph repeats itself every $360°$ **(or 2π radians)**.
  - **Amplitude:** The maximum displacement from the center. The graph oscillates between -1 and +1, so the amplitude is **1**.
  - **Domain:** All real numbers ($x \in \mathbb{R}$).
  - **Range:** $[-1,1]$ (or $-1 \leq y \leq 1$).
- **Key Points (in one cycle):**
  - $x = 0° \rightarrow y = 0$ (Starts at origin)
  - $x = 90° \rightarrow y = 1$ (Maximum)
  - $x = 180° \rightarrow y = 0$ (Crosses axis)
  - $x = 270° \rightarrow y = -1$ (Minimum)
  - $x = 360° \rightarrow y = 0$ (Returns to origin)

---

2. THE COSINE FUNCTION: $y = \cos(x)$

- **Properties:**
  - **Period:** $360°$ (or 2π radians).
  - **Amplitude:** 1.
  - **Domain:** All real numbers ($x \in \mathbb{R}$).
  - **Range:** $[-1,1]$.
- **Key Points (in one cycle):**
  - $x = 0° \rightarrow y = 1$ (Starts at maximum)
  - $x = 90° \rightarrow y = 0$ (Crosses axis)
  - $x = 180° \rightarrow y = -1$ (Minimum)
  - $x = 270° \rightarrow y = 0$ (Crosses axis)

- $x = 360° \rightarrow y = 1$ (Returns to maximum)

- **Relationship:** The cosine graph is simply the sine graph **phase-shifted** (moved) 90° to the left. $\cos(x) = \sin(x + 90°)$.

---

3. THE TANGENT FUNCTION: $y = \tan(x)$

- **Recall:** $\tan(x) = \sin(x)/\cos(x)$.
- **Properties:**
  - **Period:** The graph repeats every $180°$ **(or π radians)**.
  - **Amplitude:** Undefined (it goes to $\pm\infty$).
  - **Domain:** All real numbers *except* where $\cos(x) = 0$.
  - **Range:** All real numbers.
- **Asymptotes:** The graph has **vertical asymptotes** (lines it approaches but never touches) wherever its denominator, $\cos(x)$, is zero.
  - This occurs at $x = 90°, 270°, 450°$, etc. (i.e., $x = 90° + 180°n$).
- **Key Points (in one cycle):**
  - $x = 0° \rightarrow y = 0$
  - $x = 45° \rightarrow y = 1$
  - $x = 90° \rightarrow$ Asymptote
  - $x = 135° \rightarrow y = -1$
  - $x = 180° \rightarrow y = 0$

---

4. TRANSFORMATIONS OF TRIGONOMETRIC GRAPHS

We can modify the basic graphs using the general form: $y = A\sin(Bx + C) + D$

- A = **Amplitude:**
  - Stretches the graph vertically. $y = 3\sin(x)$ oscillates between -3 and 3.
- B = **Period Change:**

PAWILLS CONSULTING

- Changes the period (how fast it oscillates).
- New Period = $\frac{360^\circ}{B}$ (for sin/cos) or $\frac{180^\circ}{B}$ (for tan).
- Example: $y = \sin(2x)$. The period is $360/2 = 180^\circ$. The graph is "squashed" horizontally.

- C = **Phase Shift (Horizontal Shift):**
  - Moves the graph left or right. $y = \sin(x - 30^\circ)$ moves the graph 30° to the *right*.
- D = **Vertical Shift (Baseline Shift):**
  - Moves the entire graph up or down. $y = \sin(x) + 5$. The graph now oscillates between 4 and 6.
- **Worked Example 1:** Sketch $y = 2\cos(x)$ for $0^\circ \le x \le 360^\circ$.
  a. **Amplitude (A):** 2. The graph will oscillate between -2 and +2.
  b. **Period (B):** B=1. The period is still 360°.
  c. **Key Points:**
     - $x = 0^\circ \rightarrow y = 2\cos(0^\circ) = 2(1) = 2$
     - $x = 90^\circ \rightarrow y = 2\cos(90^\circ) = 2(0) = 0$
     - $x = 180^\circ \rightarrow y = 2\cos(180^\circ) = 2(-1) = -2$
     - $x = 270^\circ \rightarrow y = 2\cos(270^\circ) = 2(0) = 0$
     - $x = 360^\circ \rightarrow y = 2\cos(360^\circ) = 2(1) = 2$
  d. **Sketch:**
- **Worked Example 2:** Find the period of $y = \sin(3x)$.
  a. **Identify B:** $B = 3$.
  b. **Calculate Period:** Period = $360^\circ/B = 360^\circ/3 = 120^\circ$.
  - **Answer:** The graph completes one full cycle every 120 degrees.
- **Worked Example 3:** Sketch the graph of $y = \sin(x) - 1$ for $0^\circ \le x \le 360^\circ$.
  a. **Baseline Shift (D):** $D = -1$. The "center line" of the graph moves from $y = 0$ down to $y = -1$.

b. **Amplitude (A):** $A = 1$. The graph will go 1 unit *above* and 1 unit *below* its new center line.

c. **Range:** Max = $-1 + 1 = 0$. Min = $-1 - 1 = -2$. The graph oscillates between -2 and 0.

d. **Sketch:** Draw a normal sine wave, but shifted down 1 unit.

- $x = 0° \rightarrow y = -1$
- $x = 90° \rightarrow y = 0$ (Max)
- $x = 180° \rightarrow y = -1$
- $x = 270° \rightarrow y = -2$ (Min)
- $x = 360° \rightarrow y = -1$

EVALUATION

1. What is the **Period** and **Amplitude** of the graph $y = \sin(x)$?
2. What is the **Period** of the graph $y = \tan(x)$?
3. At what values of x (between $0°$ and $360°$) does the graph of $y = \cos(x)$ equal 0?
4. Why does the graph of $y = \tan(x)$ have vertical asymptotes?
5. Describe the transformations for the graph $y = 5\sin(2x)$. (What are its amplitude and period?)

ASSIGNMENT

1. On graph paper, carefully plot the graph of $y = 2\sin(x)$ and $y = \sin(2x)$ on separate axes, for x from $0°$ to $360°$.
2. On a separate graph paper, plot the graph of $y = \cos(x) + 1$ for x from $0°$ to $360°$. What is the new range of this function?
3. What is the amplitude and period of the function $y = -3\cos(\frac{1}{2}x)$?
4. From your graph of $y = \sin(x)$, find the approximate values of x for which $\sin(x) = 0.5$.
5. What is the equation of a cosine graph that has been shifted 3 units *up*, has an amplitude of 4, and a period of $180°$?

WEEK 13: TRIGONOMETRIC IDENTITY, INVERSE GRAPHS, & PYTHAGORAS THEOREM

1. PYTHAGORAS THEOREM (REVIEW AND APPLICATION)

- **Theorem:** In any **right-angled triangle**, the square of the hypotenuse (c) is equal to the sum of the squares of the other two sides (a and b). $a^2 + b^2 = c^2$
- **Application (3D Problems):** This is used to find diagonal lengths in 3D shapes.
- **Worked Example 1:** A cuboid (box) has length 12cm, width 5cm, and height 4cm. Find the length of the "space diagonal" AG.
    a. **Find the floor diagonal (AC):** First, find the diagonal of the base (rectangle $ABCD$). Use the triangle ABC.
        - $AC^2 = AB^2 + BC^2$
        - $AC^2 = 12^2 + 5^2 = 144 + 25 = 169$
        - $AC = \sqrt{169} = 13$ cm.
    b. **Find the space diagonal (AG):** Now use the vertical right-angled triangle ACG.
        - $AG^2 = AC^2 + CG^2$
        - $AG^2 = 13^2 + 4^2 = 169 + 16 = 185$
        - $AG = \sqrt{185}$ cm.

2. FUNDAMENTAL TRIGONOMETRIC IDENTITIES

An **identity** is an equation that is true for *all* values of the variable θ.

2A. QUOTIENT IDENTITIES

- Derived from SOH CAH TOA: $\frac{\sin(\theta)}{\cos(\theta)} = \frac{(\text{Opp}/\text{Hyp})}{(\text{Adj}/\text{Hyp})} = \frac{\text{Opp}}{\text{Hyp}} \times \frac{\text{Hyp}}{\text{Adj}} = \frac{\text{Opp}}{\text{Adj}} = \tan(\theta)$
- **Identities:** $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$ $\cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)}$

2B. PYTHAGOREAN IDENTITIES

- This is the most important set of identities.
- **The Main Identity:** Start with Pythagoras: $\text{Opp}^2 + \text{Adj}^2 = \text{Hyp}^2$
  - Divide everything by Hyp^2: $\frac{\text{Opp}^2}{\text{Hyp}^2} + \frac{\text{Adj}^2}{\text{Hyp}^2} = \frac{\text{Hyp}^2}{\text{Hyp}^2}$ $\left(\frac{\text{Opp}}{\text{Hyp}}\right)^2 + \left(\frac{\text{Adj}}{\text{Hyp}}\right)^2 = 1$
  - This gives: $\sin^2(\theta) + \cos^2(\theta) = 1$
- **The Other Two Identities:**
  - Take $\sin^2(\theta) + \cos^2(\theta) = 1$ and divide *everything* by $\cos^2(\theta)$: $\frac{\sin^2(\theta)}{\cos^2(\theta)} + \frac{\cos^2(\theta)}{\cos^2(\theta)} = \frac{1}{\cos^2(\theta)}$
  - This gives: $\tan^2(\theta) + 1 = \sec^2(\theta)$
  - Take $\sin^2(\theta) + \cos^2(\theta) = 1$ and divide *everything* by $\sin^2(\theta)$: $\frac{\sin^2(\theta)}{\sin^2(\theta)} + \frac{\cos^2(\theta)}{\sin^2(\theta)} = \frac{1}{\sin^2(\theta)}$
  - This gives: $1 + \cot^2(\theta) = \csc^2(\theta)$
- **Worked Example 2:** Prove the identity $\frac{1+\sin(x)}{\cos(x)} = \sec(x) + \tan(x)$.
  - **Strategy:** Start with the more complex side (LHS) and simplify it.
  - LHS $= \frac{1+\sin(x)}{\cos(x)}$
  - Split the fraction: $\frac{1}{\cos(x)} + \frac{\sin(x)}{\cos(x)}$
  - Use reciprocal and quotient identities: $\sec(x) + \tan(x)$
  - LHS = RHS. (Proven).
- **Worked Example 3:** Given that $\sin(x) = \frac{3}{5}$ and x is in Quadrant II, find $\cos(x)$.
  a. **Use identity:** $\sin^2(x) + \cos^2(x) = 1$
  b. **Substitute:** $(\frac{3}{5})^2 + \cos^2(x) = 1$
  c. **Solve:** $\frac{9}{25} + \cos^2(x) = 1$

- $\cos^2(x) = 1 - \frac{9}{25} = \frac{16}{25}$
- $\cos(x) = \pm\sqrt{\frac{16}{25}} = \pm\frac{4}{5}$

d. **Check Quadrant:** In Quadrant II, (ASTC), only Sine is positive. Therefore, Cosine must be **negative**.

- **Answer:** $\cos(x) = -4/5$.

---

3. GRAPHS OF INVERSE TRIGONOMETRIC RATIOS

- **What is an Inverse?**
  - $y = \sin(x)$ asks: "Here is an angle x, what is the ratio?"
  - $y = \sin^{-1}(x)$ (or $\arcsin(x)$) asks: "Here is a ratio x, what is the angle?"
- **The Problem:** The $\sin(x)$ graph fails the "Horizontal Line Test" (it repeats), so its inverse is not a true function.
- **The Solution:** We must **restrict the range** (the output) to get a "principal value."
- **Graph of $y = \arcsin(x)$:**
  - It is the graph of $y = \sin(x)$ reflected on the line $y = x$.
  - **Domain:** $[-1,1]$ (You can only find arcsin of numbers between -1 and 1).
  - **Range (Principal Value):** $[-90°, 90°]$ or $[-\pi/2, \pi/2]$.
- **Graph of $y = \arccos(x)$:**
  - **Domain:** $[-1,1]$
  - **Range (Principal Value):** $[0°, 180°]$ or $[0, \pi]$.
- **Graph of $y = \arctan(x)$:**
  - **Domain:** All real numbers.
  - **Range (Principal Value):** $(-90°, 90°)$ or $(-\pi/2, \pi/2)$. (Has horizontal asymptotes).
- **Worked Example 4:** Find the principal value of $\arcsin(-\frac{1}{2})$.

a. **Question:** "The sine of *what angle* gives $-\frac{1}{2}$?"

b. **Reference Angle:** We know $\sin(30°) = \frac{1}{2}$.

c. **Quadrant:** Sine is negative in Q.III and Q.IV.

d. **Range:** The principal value range for arcsin is $[-90°, 90°]$.

e. **Solution:** The angle in Q.IV that is within this range is $-30°$.

- **Answer:** $\arcsin(-\frac{1}{2}) = -30°$.

EVALUATION

1. A cuboid has dimensions 8cm, 6cm, and 5cm. Find the length of its space diagonal.
2. State the three Pythagorean identities.
3. Prove the identity: $\sin(x) \times \csc(x) = 1$.
4. What is the "principal value" of an inverse trigonometric function?
5. Find the principal value (in degrees) of (a) $\arcsin(\frac{\sqrt{3}}{2})$ and (b) $\arccos(-\frac{1}{2})$.

ASSIGNMENT

1. Prove the identity: $\frac{1}{\tan(\theta)+\cot(\theta)} = \sin(\theta)\cos(\theta)$.
2. If $\sin(x) = 4/5$ and x is in Quadrant II, find the values of $\cos(x)$ and $\tan(x)$ using identities.
3. Sketch the graph of $y = \arccos(x)$, stating its domain and range.
4. Find the principal value (in degrees) of (a) $\arctan(1)$ and (b) $\arcsin(-1)$.
5. A right-angled triangle has sides 5, 12, and 13. Let θ be the angle opposite the side of length 5. Prove that $\sin^2(\theta) + \cos^2(\theta) = 1$ using the side lengths.

SS2 FURTHER MATHEMATICS LESSON NOTES

SECOND TERM

WEEK 1: LIMITS OF FUNCTIONS AND DIFFERENTIATION FROM FIRST PRINCIPLE

1. CONCEPT OF LIMITS

The concept of a **limit** is the foundation of calculus (which includes differentiation and integration). A limit is a value that a function $f(x)$ "approaches" as the input x "approaches" a certain value.

Intuitive Definition: We say $\lim_{x \to a} f(x) = L$ ("the limit of $f(x)$ as x approaches a is L") if we can make the value of $f(x)$ arbitrarily close to L by taking x sufficiently close to a, but **not equal** to a.

The limit is concerned with the *behavior* of the function *near* a point, not *at* the point itself.

2. EVALUATION OF LIMITS

There are several methods to find the limit of a function.

2A. DIRECT SUBSTITUTION

This is the first method you should always try. If the function is "well-behaved" (i.e., it is *continuous* at $x = a$), you can find the limit by simply plugging in the value a.

- **Worked Example 1:** Find $\lim_{x \to 2}(3x^2 - 5x + 1)$
    - This is a polynomial, which is continuous everywhere. We can substitute $x = 2$ directly.
    - $\lim_{x \to 2}(3x^2 - 5x + 1) = 3(2)^2 - 5(2) + 1 = 3(4) - 10 + 1 = 12 - 10 + 1 = 3$.

2B. INDETERMINATE FORMS (0/0) - FACTORIZATION

Sometimes, direct substitution results in an **indeterminate form** like $\frac{0}{0}$. This *does not* mean the limit is 0 or undefined. It means we must do more algebraic work. The most common method is factorization.

- **Worked Example 2:** Find $\lim_{x\to 3}\frac{x^2-9}{x-3}$
    a. **Test:** Substituting $x = 3$ gives $\frac{3^2-9}{3-3} = \frac{9-9}{3-3} = \frac{0}{0}$. (Indeterminate form).
    b. **Factorize:** The numerator is a difference of two squares. $\lim_{x\to 3}\frac{(x-3)(x+3)}{x-3}$
    c. **Cancel:** Since x is *approaching* 3 but *not equal* to 3, $(x-3)$ is not zero, so we can cancel the common factor. $\lim_{x\to 3}(x+3)$
    d. **Evaluate:** Now we can use direct substitution. $3+3=6$
    - **Answer:** $\lim_{x\to 3}\frac{x^2-9}{x-3} = 6$.

2C. INDETERMINATE FORMS (0/0) - RATIONALIZATION (USING CONJUGATES)

If the $\frac{0}{0}$ form involves a square root, we must rationalize the numerator or denominator by multiplying by its conjugate.

- **Worked Example 3:** Find $\lim_{x\to 0}\frac{\sqrt{x+4}-2}{x}$
    e. **Test:** Substituting $x = 0$ gives $\frac{\sqrt{0+4}-2}{0} = \frac{2-2}{0} = \frac{0}{0}$. (Indeterminate form).
    f. **Rationalize:** The conjugate of $(\sqrt{x+4}-2)$ is $(\sqrt{x+4}+2)$. $\lim_{x\to 0}\left[\frac{\sqrt{x+4}-2}{x}\right]\cdot\left[\frac{\sqrt{x+4}+2}{\sqrt{x+4}+2}\right]$
    g. **Simplify (Numerator):** $(a-b)(a+b) = a^2-b^2$ $\lim_{x\to 0}\frac{(\sqrt{x+4})^2-(2)^2}{x(\sqrt{x+4}+2)}$
    $\lim_{x\to 0}\frac{(x+4)-4}{x(\sqrt{x+4}+2)}$ $\lim_{x\to 0}\frac{x}{x(\sqrt{x+4}+2)}$
    h. **Cancel:** Cancel the x term. $\lim_{x\to 0}\frac{1}{\sqrt{x+4}+2}$
    i. **Evaluate:** Now substitute $x = 0$. $\frac{1}{\sqrt{0+4}+2} = \frac{1}{\sqrt{4}+2} = \frac{1}{2+2} = \frac{1}{4}$

- **Answer:** $\lim_{x\to 0}\frac{\sqrt{x+4}-2}{x}=\frac{1}{4}$.

---

3. DIFFERENTIATION USING FIRST PRINCIPLE

Differentiation is the process of finding the **derivative** of a function. The derivative, written as $f'(x)$ or $\frac{dy}{dx}$, represents the **instantaneous rate of change** or the **gradient of the tangent** to the curve at any point x.

The **First Principle** is the formal *definition* of the derivative, derived from the concept of limits. It finds the gradient of a secant line between two points $(x, f(x))$ and $(x+h, f(x+h))$ and then finds the limit as the distance h between them shrinks to zero.

The Formula (First Principle): $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

- **Worked Example 4:** Find the derivative of $f(x)=x^2$ from the first principle.
  j. **Identify $f(x)$ and $f(x+h)$:**
    - $f(x)=x^2$
    - $f(x+h)=(x+h)^2=x^2+2xh+h^2$
  k. **Substitute into the formula:** $f'(x)=\lim_{h\to 0}\frac{(x^2+2xh+h^2)-(x^2)}{h}$
  l. **Simplify the numerator:** $f'(x)=\lim_{h\to 0}\frac{x^2-x^2+2xh+h^2}{h}$ $f'(x)=\lim_{h\to 0}\frac{2xh+h^2}{h}$
  m. **Factor out h from the numerator:** $f'(x)=\lim_{h\to 0}\frac{h(2x+h)}{h}$
  n. **Cancel h:** $f'(x)=\lim_{h\to 0}(2x+h)$
  o. **Evaluate the limit (substitute $h=0$):** $f'(x)=2x+0=2x$
  - **Answer:** The derivative of x^2 is $2x$.
- **Worked Example 5:** Find the derivative of $f(x)=3x+2$ from the first principle.
  p. **Identify $f(x)$ and $f(x+h)$:**
    - $f(x)=3x+2$
    - $f(x+h)=3(x+h)+2=3x+3h+2$

q. **Substitute into the formula:** $f'(x) = \lim_{h\to 0} \frac{(3x+3h+2)-(3x+2)}{h}$

r. **Simplify the numerator:** $f'(x) = \lim_{h\to 0} \frac{3x-3x+3h+2-2}{h}$ $f'(x) = \lim_{h\to 0} \frac{3h}{h}$

s. **Cancel h:** $f'(x) = \lim_{h\to 0} 3$

t. **Evaluate the limit:** $f'(x) = 3$

- **Answer:** The derivative of $3x + 2$ is **3**. (This makes sense, as the gradient of the straight line $y = 3x + 2$ is always 3).

EVALUATION

1. Explain the intuitive meaning of $\lim_{x\to a} f(x) = L$.
2. Evaluate $\lim_{x\to -1}(x^3 - 4x^2 + 2)$.
3. Evaluate $\lim_{x\to 5} \frac{x^2-25}{x-5}$.
4. Evaluate $\lim_{x\to 2} \frac{x^2-5x+6}{x^2-4}$.
5. Use the first principle to find the derivative of the function $f(x) = x^2 - x$.

ASSIGNMENT

1. Evaluate $\lim_{x\to 4} \frac{\sqrt{x}-2}{x-4}$. (Hint: Rationalize the numerator).
2. Use the first principle to find the derivative of $f(x) = 2x^2 + 5x$.
3. Use the first principle to find the derivative of $f(x) = x^3$. (Hint: $(x + h)^3 = x^3 + 3x^2h + 3xh^2 + h^3$).
4. Use the first principle to find the derivative of $f(x) = \frac{1}{x}$. (Hint: $f(x + h) - f(x) = \frac{1}{x+h} - \frac{1}{x}$. Find a common denominator).
5. Define "indeterminate form" and provide two examples.

WEEK 2: RULES OF DIFFERENTIATION I

1. INTRODUCTION

Last week, we found the derivative using the "First Principle." This method is long and tedious. This week, we learn the "shortcuts" or **rules** that allow us to find the derivative of algebraic functions quickly.

---

2. THE CONSTANT RULE

If $f(x) = c$ (where c is a constant), then $f'(x) = 0$.

- **Meaning:** The graph of $y = c$ is a horizontal line. Its slope (gradient) is always zero.
- **Example:** If $y = 10$, $\frac{dy}{dx} = 0$.
- **Example:** If $y = \pi$, $\frac{dy}{dx} = 0$.

---

3. THE POWER RULE

This is the most important rule in all of calculus. If $f(x) = x^n$ (where n is any real number), then $f'(x) = nx^{n-1}$.

- **Mnemonic:** "Bring down the power, then reduce the power by one."
- **Worked Example 1:** $y = x^5$
    - $\frac{dy}{dx} = 5x^{5-1} = 5x^4$.
- **Worked Example 2 (Fractional Exponents):** $y = \sqrt{x}$
    a. **Rewrite:** $\sqrt{x} = x^{1/2}$.
    b. **Differentiate:** $\frac{dy}{dx} = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{-1/2}$
    c. **Simplify:** $\frac{dy}{dx} = \frac{1}{2} \cdot \frac{1}{x^{1/2}} = \frac{1}{2\sqrt{x}}$.

PAWELS CONSULTING

- **Worked Example 3 (Negative Exponents):** $y = \frac{1}{x^3}$

    d. **Rewrite:** $\frac{1}{x^3} = x^{-3}$.

    e. **Differentiate:** $\frac{dy}{dx} = -3x^{-3-1} = -3x^{-4}$

    f. **Simplify:** $\frac{dy}{dx} = -\frac{3}{x^4}$.

---

4. THE CONSTANT MULTIPLE RULE

If $y = c \cdot f(x)$ (where c is a constant), then $\frac{dy}{dx} = c \cdot f'(x)$.

- **Meaning:** Constants "just tag along." You differentiate the function and multiply the result by the constant.
- **Worked Example 4:** $y = 7x^4$

    g. **Differentiate x^4:** $4x^3$

    h. **Multiply by constant:** $\frac{dy}{dx} = 7 \cdot (4x^3) = 28x^3$.

---

5. THE SUM/DIFFERENCE RULE

If $y = f(x) \pm g(x)$, then $\frac{dy}{dx} = f'(x) \pm g'(x)$.

- **Meaning:** You can differentiate a polynomial *term by term*.
- **Worked Example 5 (Full Polynomial):** Find the derivative of $y = 4x^3 - 5x^2 + 7x - 10$.
    - We differentiate each term separately.
    - $d/dx(4x^3) = 4 \cdot (3x^2) = 12x^2$
    - $d/dx(-5x^2) = -5 \cdot (2x^1) = -10x$
    - $d/dx(7x) = d/dx(7x^1) = 7 \cdot (1x^0) = 7 \cdot (1) = 7$
    - $d/dx(-10) = 0$ (Constant Rule)

- **Answer:** $\frac{dy}{dx} = 12x^2 - 10x + 7$.

- **Worked Example 6 (Complex Expression):** Find $\frac{dy}{dx}$ for $y = 2\sqrt{x} - \frac{5}{x^2} + \pi^3$.

  i. **Rewrite the function** using powers:

  - $y = 2x^{1/2} - 5x^{-2} + \pi^3$ (Note: π^3 is just a constant!)

  j. **Differentiate term by term:**

  - $d/dx(2x^{1/2}) = 2 \cdot (\frac{1}{2}x^{\frac{1}{2}-1}) = 1x^{-1/2}$
  - $d/dx(-5x^{-2}) = -5 \cdot (-2x^{-2-1}) = +10x^{-3}$
  - $d/dx(\pi^3) = 0$

  k. **Combine:** $\frac{dy}{dx} = x^{-1/2} + 10x^{-3}$

  l. **Simplify (optional but good):** $\frac{dy}{dx} = \frac{1}{\sqrt{x}} + \frac{10}{x^3}$.

- **Worked Example 7 (Simplify First):** Find $f'(x)$ for $f(x) = \frac{4x^4 - 3x^2}{x}$.

  - **Do NOT** use the Quotient Rule (Week 3) yet. Simplify first!
  - $f(x) = \frac{4x^4}{x} - \frac{3x^2}{x} = 4x^3 - 3x$.
  - Now, differentiate:
  - $f'(x) = 4(3x^2) - 3(1x^0) = 12x^2 - 3$.

EVALUATION

1. What is the derivative of a constant, $y = 100$? Why?
2. State the Power Rule for $y = x^n$.
3. Find $\frac{dy}{dx}$ if $y = 7x^5 - 2x^4 + x - 9$.
4. Differentiate $f(x) = 6\sqrt{x}$.
5. Differentiate $g(x) = \frac{8}{x} + \frac{1}{2x^2}$. (Hint: Rewrite first).

PAVELLS CONSULTING

ASSIGNMENT

1. Find the derivative of $y = (2x + 3)(x - 1)$. (Hint: Expand the brackets *first*, then differentiate).
2. Differentiate $f(t) = 4t^3 - \sqrt[3]{t} + 2$. (Hint: $\sqrt[3]{t} = t^{1/3}$).
3. Differentiate $y = \frac{5x^3+x^2-6}{x^2}$. (Hint: Simplify first by dividing each term by x^2).
4. Find the gradient of the tangent to the curve $y = x^3 - 2x^2 + 5$ at the point $x = 2$. (This is dy/dx at $x = 2$).
5. Find the derivative of $y = a^2x^3 - bx + c^4$. (Assume a, b, c are constants).

PAWILLS CONSULTING

WEEK 3: RULES OF DIFFERENTIATION II

1. INTRODUCTION

Last week, we handled sums and differences of simple functions. But what happens if we need to differentiate a *product* of two functions, or a *quotient* (division)? We cannot simply differentiate them separately. We must use specific rules.

---

2. THE PRODUCT RULE

Used to differentiate the *product* of two functions. Let $y = u \cdot v$, where $u = f(x)$ and $v = g(x)$.

The Rule: $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$ In prime notation: $(uv)' = u'v + uv'$ (Note: the order doesn't matter, $u'v + v'u$ is also common)

- **Mnemonic:** "Differentiate the first, keep the second, PLUS, keep the first, differentiate the second."
- **Worked Example 1:** Find $\frac{dy}{dx}$ for $y = (x^2 + 1)(3x - 2)$.
    a. **Identify u and v:**
        - $u = (x^2 + 1)$
        - $v = (3x - 2)$
    b. **Find their derivatives (u' and v'):**
        - $\frac{du}{dx} = 2x$
        - $\frac{dv}{dx} = 3$
    c. **Apply the formula $\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$:**
        - $\frac{dy}{dx} = (3x - 2) \cdot (2x) + (x^2 + 1) \cdot (3)$
    d. **Simplify (expand and collect terms):**

- $\frac{dy}{dx} = (6x^2 - 4x) + (3x^2 + 3)$
- $\frac{dy}{dx} = 9x^2 - 4x + 3$

- **Worked Example 2:** Differentiate $y = x^3 \cdot (4x^2 - 5)$.

e. **Identify u and v:** $u = x^3$, $v = 4x^2 - 5$

f. **Find u' and v':** $u' = 3x^2$, $v' = 8x$

g. **Apply formula ($u'v + uv'$):**

- $\frac{dy}{dx} = (3x^2) \cdot (4x^2 - 5) + (x^3) \cdot (8x)$

h. **Simplify:**

- $\frac{dy}{dx} = (12x^4 - 15x^2) + (8x^4)$
- $\frac{dy}{dx} = 20x^4 - 15x^2$

---

3. THE QUOTIENT RULE

Used to differentiate the *quotient* (division) of two functions. Let $y = \frac{u}{v}$, where $u = f(x)$ and $v = g(x)$.

The Rule: $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$ In prime notation: $\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$ (Note: many use $\frac{vu' - uv'}{v^2}$, which is the same. The order of subtraction *is* critical).

- **Mnemonic:** "Low d-High minus High d-Low, all over Low-Low."
  - (Low = v, High = u, d-High = u', d-Low = v') $\Rightarrow \frac{v \cdot u' - u \cdot v'}{v^2}$

- **Worked Example 3:** Find $\frac{dy}{dx}$ for $y = \frac{x^2 - 1}{x + 3}$.

i. **Identify u and v:**

- $u = x^2 - 1$ (High)
- $v = x + 3$ (Low)

j. **Find u' and v':**

- $u' = 2x$ (d-High)
- $v' = 1$ (d-Low)

k. **Apply the formula $\frac{vu'-uv'}{v^2}$:**

- $\frac{dy}{dx} = \frac{(x+3)(2x)-(x^2-1)(1)}{(x+3)^2}$

l. **Simplify the *numerator*:**

- $\frac{dy}{dx} = \frac{(2x^2+6x)-(x^2-1)}{(x+3)^2}$
- $\frac{dy}{dx} = \frac{2x^2+6x-x^2+1}{(x+3)^2}$
- $\frac{dy}{dx} = \frac{x^2+6x+1}{(x+3)^2}$

- **Note:** Do *not* expand the denominator. Leave it factored.

---

4. THE CHAIN RULE (FUNCTION OF A FUNCTION)

Used to differentiate a *composite* function (a function *inside* another function). Let $y = f(g(x))$. Let $u = g(x)$, so $y = f(u)$.

The Rule: $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

- **Mnemonic:** "Differentiate the *outside* function (leaving the inside alone), THEN, multiply by the derivative of the *inside* function."
- **Worked Example 4 (General Power Rule):** $y = (2x^2 + 5)^4$

m. **Identify "outside" and "inside":**

- Outside function: u^4
- Inside function (u): $2x^2 + 5$

n. **Differentiate the *outside*:**

- $d/du(u^4) = 4u^3 \Rightarrow 4(2x^2 + 5)^3$

o. **Differentiate the *inside*:**

- $d/dx(2x^2 + 5) = 4x$

p. **Multiply them together:**

- $\frac{dy}{dx} = [4(2x^2 + 5)^3] \cdot [4x]$
- $\frac{dy}{dx} = 16x(2x^2 + 5)^3$

- **Worked Example 5 (Root):** $y = \sqrt{x^3 - 1}$

q. **Rewrite:** $y = (x^3 - 1)^{1/2}$

r. **Identify:**

- Outside: $u^{1/2}$
- Inside (u): $x^3 - 1$

s. **Differentiate *outside*:**

- $\frac{1}{2}u^{-1/2} \Rightarrow \frac{1}{2}(x^3 - 1)^{-1/2}$

t. **Differentiate *inside*:**

- $3x^2$

u. **Multiply:**

- $\frac{dy}{dx} = [\frac{1}{2}(x^3 - 1)^{-1/2}] \cdot [3x^2]$
- $\frac{dy}{dx} = \frac{3x^2}{2(x^3-1)^{1/2}} = \frac{3x^2}{2\sqrt{x^3-1}}$

EVALUATION

1. State the Product Rule.
2. State the Quotient Rule.
3. State the Chain Rule (in dy/dx notation).
4. Find the derivative of $y = (x^3 + 4)(2x - 1)$ using the product rule.
5. Find the derivative of $y = \frac{x}{x^2+1}$ using the quotient rule.

ASSIGNMENT

1. Differentiate $f(x) = (3x^2 - 1)^5$ using the chain rule.
2. Differentiate $y = \frac{2x-3}{4x+5}$ using the quotient rule.

3. Find the derivative of $y = (x^2 + 3x)\sqrt{x}$. (Hint: This is a product rule, or you can simplify first: $(x^2 + 3x)x^{1/2} = x^{5/2} + 3x^{3/2}$).
4. Find the equation of the tangent line to the curve $y = \frac{x-1}{x+1}$ at the point $x = 1$.
5. **Challenge:** Find the derivative of $y = \frac{x}{(x^2+1)^3}$. (Hint: This requires both the quotient rule and the chain rule).

WEEK 4: DIFFERENTIATION OF TRIGONOMETRIC, LOGARITHMIC, AND EXPONENTIAL FUNCTIONS

1. INTRODUCTION

This week, we expand our "differentiation toolkit" to include non-algebraic (transcendental) functions. The key is to **memorize the standard derivatives** and then **combine them with the rules** (Product, Quotient, Chain) from last week.

---

2. STANDARD DERIVATIVES (TO MEMORIZE)

2A. TRIGONOMETRIC FUNCTIONS

- $\frac{d}{dx}(\sin x) = \cos x$
- $\frac{d}{dx}(\cos x) = -\sin x$
- $\frac{d}{dx}(\tan x) = \sec^2 x$
- **Reciprocal Functions (Further Maths):**
  - $\frac{d}{dx}(\sec x) = \sec(x)\tan(x)$
  - $\frac{d}{dx}(\csc x) = -\csc(x)\cot(x)$
  - $\frac{d}{dx}(\cot x) = -\csc^2(x)$

2B. EXPONENTIAL & LOGARITHMIC FUNCTIONS

- **The Natural Exponential Function (Euler's Number e):**
  - $\frac{d}{dx}(e^x) = e^x$ (This is the *only* function that is its own derivative!)
- **The Natural Logarithm:**
  - $\frac{d}{dx}(\ln x) = \frac{1}{x}$ (for $x > 0$)
- **Other Bases (Further Maths):**

- $\frac{d}{dx}(a^x) = a^x \ln(a)$
- $\frac{d}{dx}(\log_a x) = \frac{1}{x\ln(a)}$

---

3. APPLICATION TO PROBLEMS (COMBINING WITH CHAIN RULE)

The most common application is to combine these new functions with the **Chain Rule**. **Chain Rule (Mnemonic):** "Derivative of the *outside* (respecting the inside) × Derivative of the *inside*."

3A. WORKED EXAMPLES (TRIGONOMETRIC)

- **Worked Example 1:** Differentiate $y = \sin(5x)$.
  a. **Outside:** $\sin(u) \Rightarrow$ derivative is $\cos(u)$.
  b. **Inside:** $u = 5x \Rightarrow$ derivative is 5.
  c. **Chain Rule:** $\frac{dy}{dx} = [\cos(5x)] \cdot [5]$
  - **Answer:** $\frac{dy}{dx} = 5\cos(5x)$
- **Worked Example 2:** Differentiate $y = \cos(x^3)$.
  d. **Outside:** $\cos(u) \Rightarrow$ derivative is $-\sin(u)$.
  e. **Inside:** $u = x^3 \Rightarrow$ derivative is $3x^2$.
  f. **Chain Rule:** $\frac{dy}{dx} = [-\sin(x^3)] \cdot [3x^2]$
  - **Answer:** $\frac{dy}{dx} = -3x^2\sin(x^3)$
- **Worked Example 3:** Differentiate $y = \tan^4(x)$.
  g. **Rewrite:** $y = [\tan(x)]^4$. This is a power rule problem.
  h. **Outside:** $u^4 \Rightarrow$ derivative is $4u^3$.
  i. **Inside:** $u = \tan(x) \Rightarrow$ derivative is $\sec^2(x)$.
  j. **Chain Rule:** $\frac{dy}{dx} = [4(\tan(x))^3] \cdot [\sec^2(x)]$

LAWELL CONSULTING

- **Answer:** $\frac{dy}{dx} = 4\tan^3(x)\sec^2(x)$

3B. WORKED EXAMPLES (EXPONENTIAL & LOGARITHMIC)

- **Worked Example 4:** Differentiate $y = e^{7x-2}$.
    k. **Outside:** $e^u \Rightarrow$ derivative is e^u.
    l. **Inside:** $u = 7x - 2 \Rightarrow$ derivative is 7.
    m. **Chain Rule:** $\frac{dy}{dx} = [e^{7x-2}] \cdot [7]$
    - **Answer:** $\frac{dy}{dx} = 7e^{7x-2}$
- **Worked Example 5:** Differentiate $y = \ln(x^2 - 4x)$.
    n. **Outside:** $\ln(u) \Rightarrow$ derivative is $1/u$.
    o. **Inside:** $u = x^2 - 4x \Rightarrow$ derivative is $2x - 4$.
    p. **Chain Rule:** $\frac{dy}{dx} = \left[\frac{1}{x^2-4x}\right] \cdot [2x - 4]$
    - **Answer:** $\frac{dy}{dx} = \frac{2x-4}{x^2-4x}$

3C. WORKED EXAMPLES (MIXED RULES)

- **Worked Example 6 (Product Rule):** Differentiate $y = x^2\sin(x)$.
    q. **Identify u and v:** $u = x^2$, $v = \sin(x)$
    r. **Find u' and v':** $u' = 2x$, $v' = \cos(x)$
    s. **Apply Product Rule ($u'v + uv'$):**
        - $\frac{dy}{dx} = (2x) \cdot (\sin x) + (x^2) \cdot (\cos x)$
    - **Answer:** $\frac{dy}{dx} = 2x\sin(x) + x^2\cos(x)$
- **Worked Example 7 (Quotient Rule):** Differentiate $y = \frac{e^{3x}}{x^2}$.
    t. **Identify u and v:** $u = e^{3x}$, $v = x^2$
    u. **Find u' and v':**

- $u' = e^{3x} \cdot 3 = 3e^{3x}$ (Chain Rule)
- $v' = 2x$

v. **Apply Quotient Rule ($\frac{vu' - uv'}{v^2}$):**

- $\frac{dy}{dx} = \frac{(x^2)(3e^{3x}) - (e^{3x})(2x)}{(x^2)^2}$

w. **Simplify (Factor out xe^{3x}):**

- $\frac{dy}{dx} = \frac{xe^{3x}(3x-2)}{x^4}$

- **Answer:** $\frac{dy}{dx} = \frac{e^{3x}(3x-2)}{x^3}$

EVALUATION

1. Find the derivative of $y = 5\cos(x) - 2\sin(x)$.
2. Find $\frac{dy}{dx}$ if $y = \tan(7x)$.
3. Differentiate $f(x) = e^{x^2}$.
4. Differentiate $g(x) = \ln(x^3 + 1)$.
5. Find the derivative of $y = x \cdot \ln(x)$ (using the product rule).

ASSIGNMENT

1. Find the derivative of $y = \sin^3(x)$. (Hint: $y = [\sin(x)]^3$).
2. Find $\frac{dy}{dx}$ if $y = \frac{\sin(x)}{x}$.
3. Find the derivative of $y = e^x\cos(x)$ (using the product rule).
4. Find the equation of the tangent line to the curve $y = e^{2x}$ at the point $x = 0$.
5. Differentiate $y = \sec(4x^2)$.

WEEK 5: MIDTERM EXAMINATION

This week is dedicated to the Midterm Examination, which will cover all topics from Weeks 1–4.

- **Topics:**
  - **Week 1:** Limits (Direct Substitution, Factorization, Rationalization), Differentiation from First Principle.
  - **Week 2:** Rules of Differentiation (Power, Constant, Sum/Difference), including negative and fractional exponents.
  - **Week 3:** Advanced Rules (Product, Quotient, Chain) for algebraic functions.
  - **Week 4:** Standard Derivatives (Trig, e^x, $\ln x$) and their combination with the Chain, Product, and Quotient rules.
- **Format:** The test will include definitions, finding limits, differentiation from first principle, and a wide range of differentiation problems using all the rules.

PAWILLS CONSULTING

WEEK 6: APPLICATION OF DIFFERENTIATION

1. RATE OF CHANGE

The derivative $\frac{dy}{dx}$ literally means "the rate of change of y with respect to x." If $A = \pi r^2$ (Area of a circle), then $\frac{dA}{dr} = 2\pi r$ represents the *instantaneous rate of change of the Area with respect to the Radius*.

- **Worked Example 1:** The area A of a square is related to its side s by the formula $A = s^2$. Find the rate of change of the Area with respect to the side when the side is 5cm.
    a. **Find the derivative:** $\frac{dA}{ds} = 2s$.
    b. **Evaluate at** $s = 5$**:** $\frac{dA}{ds} = 2(5) = 10$.
    - **Answer:** The rate of change is 10 cm^2/cm. (The area is increasing at a rate of 10 cm^2 per 1 cm increase in the side length).
- **Related Rates (Chain Rule):** Often, we know how one variable changes *with time* ($\frac{dx}{dt}$) and we want to find how another variable changes *with time* ($\frac{dy}{dt}$). We use the chain rule.
  $\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$
- **Worked Example 2:** The radius of a circle is increasing at a rate of 2 cm/s. Find the rate at which the area is increasing when the radius is 10 cm.
    a. **Identify knowns:**
        - We know $\frac{dr}{dt} = 2$ cm/s.
        - We want to find $\frac{dA}{dt}$ at the instant when $r = 10$ cm.
    b. **Find the relationship:** $A = \pi r^2$.
    c. **Differentiate *with respect to time (t)*:**
        - $\frac{d}{dt}(A) = \frac{d}{dt}(\pi r^2)$
        - We must use the chain rule on the right side:

- $\frac{dA}{dt} = \frac{dA}{dr} \cdot \frac{dr}{dt}$
- $\frac{dA}{dt} = (2\pi r) \cdot \frac{dr}{dt}$

d. **Substitute the known values:**

- $\frac{dA}{dt} = (2\pi \cdot 10) \cdot (2)$
- $\frac{dA}{dt} = 20\pi \cdot 2 = 40\pi$

- **Answer:** The area is increasing at a rate of $40\pi\ cm^2/s$.

---

2. VELOCITY AND ACCELERATION (KINEMATICS)

This is the most direct application of differentiation. We analyze the motion of a particle.

- **Displacement (s):** The position (distance from origin) of the particle at time t.
- **Velocity (v):** The rate of change of displacement. It is the *derivative* of displacement. $v(t) = \frac{ds}{dt}$
- **Acceleration (a):** The rate of change of velocity. It is the *derivative* of velocity. $a(t) = \frac{dv}{dt}$
- **Second Derivative:** This means acceleration is also the *second derivative* of displacement. $a(t) = \frac{d^2s}{dt^2}$

Key Concepts:

- When a particle is **"at rest"** or **"stationary"**, its velocity is zero. $\mathbf{v(t) = 0}$.
- When velocity is **positive** ($v > 0$), the particle is moving in the positive direction (e.g., forward/up).
- When velocity is **negative** ($v < 0$), the particle is moving in the negative direction (e.g., backward/down).
- **Worked Example 3:** A particle's displacement (in meters) along the x-axis is given by the equation of motion $s(t) = t^3 - 6t^2 + 9t + 1$, for $t \geq 0$.

- **(a) Find the velocity and acceleration functions.**
    i. **Velocity ($v = ds/dt$):** $v(t) = 3t^2 - 12t + 9$
    ii. **Acceleration ($a = dv/dt$):** $a(t) = 6t - 12$
- **(b) Find the initial velocity.**
    - "Initial" means at time $t = 0$.
    - $v(0) = 3(0)^2 - 12(0) + 9 = 9$ m/s.
- **(c) At what time(s) is the particle momentarily at rest?**
    iii. "At rest" means $v(t) = 0$.
    iv. $3t^2 - 12t + 9 = 0$
    v. Divide by 3: $t^2 - 4t + 3 = 0$
    vi. Factorize: $(t - 1)(t - 3) = 0$
    - **Answer:** The particle is at rest at $\mathbf{t = 1s}$ and $\mathbf{t = 3s}$.
- **(d) What is the acceleration when the particle is at rest?**
    - At $t = 1$: $a(1) = 6(1) - 12 = -6$ m/s^2.
    - At $t = 3$: $a(3) = 6(3) - 12 = 18 - 12 = 6$ m/s^2.

EVALUATION

1. If $V = \frac{4}{3}\pi r^3$ (volume of a sphere), find $\frac{dV}{dr}$. What does this represent?
2. The side of a square, s, is increasing at a rate of 3 cm/s. Find the rate at which the area, A, is increasing when the side is 10 cm.
3. Given the displacement of a particle $s(t) = 2t^3 - 9t^2 + 12t - 1$.
    - (a) Find the velocity function $v(t)$.
    - (b) Find the acceleration function $a(t)$.
4. For the particle in Q3, what is its velocity at $t = 2s$?
5. For the particle in Q3, at what time(s) is the particle "at rest"?

PAWILEN CONSULTING

ASSIGNMENT

1. A spherical balloon is being inflated. Its radius r is increasing at a rate of 0.5 cm/s. Find the rate at which its volume V is increasing when the radius is 4 cm. (Given $V = \frac{4}{3}\pi r^3$).
2. The displacement of a particle is given by $s(t) = t^3 - 3t^2 - 9t + 5$.
    - (a) Find the velocity and acceleration as functions of t.
    - (b) Find the time(s) when the particle is stationary.
    - (c) Find the displacement of the particle when its acceleration is zero.
3. If a ball is thrown upwards with its height (in meters) given by $h(t) = 40 + 30t - 5t^2$,
    - (a) What is its initial velocity?
    - (b) What is its velocity at $t = 2s$?
    - (c) At what time does the ball reach its maximum height? (Hint: Max height is when $v(t) = 0$).
4. Find the rate of change of the area of a circle with respect to its circumference. (Hint: $A = \pi r^2$ and $C = 2\pi r$. Find $\frac{dA}{dC} = \frac{dA/dr}{dC/dr}$).
5. Define: Displacement, Velocity, Acceleration, Second Derivative.

PAWILLS CONSULTING

WEEK 7: MIDTERM BREAK

This week is for the Midterm Break. There are no formal lesson notes.

PAWILLS CONSULTING

WEEK 8: MAXIMUM AND MINIMUM POINTS AND VALUES

1. TURNING POINTS OF CURVES

Turning points (also called **stationary points**) on a curve $y = f(x)$ are points where the curve "turns" from rising to falling, or from falling to rising.

- At these points, the gradient of the tangent line is zero (it is a horizontal line).
- Therefore, the condition for finding a stationary point is to set the first derivative to zero: $\frac{dy}{dx} = 0$

There are three types of stationary points:

1. **Local Maximum:** The curve is at a "peak." The gradient goes from positive → zero → negative.
2. **Local Minimum:** The curve is at a "valley." The gradient goes from negative → zero → positive.
3. **Point of Inflexion:** The curve "flattens out" but does not turn. (e.g., $y = x^3$ at $x = 0$).

---

2. THE SECOND DERIVATIVE TEST (FOR CLASSIFICATION)

Once we find the x-values of the stationary points (by solving $dy/dx = 0$), we must *classify* them. The easiest way is the **Second Derivative Test**.

- The **Second Derivative,** $f''(x)$ **or** $\frac{d^2y}{dx^2}$, tells us the *rate of change of the gradient*. It measures the **concavity** of the curve.
    - If $\frac{d^2y}{dx^2} > 0$ (positive), the gradient is *increasing*. The curve is "concave up" (shaped like a "U"). This indicates a **LOCAL MINIMUM**.
    - If $\frac{d^2y}{dx^2} < 0$ (negative), the gradient is *decreasing*. The curve is "concave down" (shaped like an "∩"). This indicates a **LOCAL MAXIMUM**.

- If $\frac{d^2y}{dx^2} = 0$, the test is inconclusive. You must use the First Derivative Test (checking signs).

Steps to find and classify turning points:

1. Find the first derivative, $\frac{dy}{dx}$.
2. Find the second derivative, $\frac{d^2y}{dx^2}$.
3. Set $\frac{dy}{dx} = 0$ and solve for x to find the stationary points.
4. Substitute these x-values into $\frac{d^2y}{dx^2}$ to classify them.
5. Substitute the x-values into the *original* equation $y = f(x)$ to find the y-coordinates of the points.

- **Worked Example 1:** Find and classify the turning points of the curve $y = 2x^3 - 3x^2 - 12x + 1$.
  a. **First Derivative:**
     - $\frac{dy}{dx} = 6x^2 - 6x - 12$
  b. **Second Derivative:**
     - $\frac{d^2y}{dx^2} = 12x - 6$
  c. **Find Stationary Points (set $dy/dx = 0$):**
     - $6x^2 - 6x - 12 = 0$
     - $x^2 - x - 2 = 0$
     - $(x - 2)(x + 1) = 0$
     - Stationary points are at $\mathbf{x = 2}$ and $\mathbf{x = -1}$.
  d. **Classify points (using $d^2y/dx^2 = 12x - 6$):**
     - **At $x = 2$:** $\frac{d^2y}{dx^2} = 12(2) - 6 = 18$.
       - Since $18 > 0$ (positive), this is a **Local Minimum**.
     - **At $x = -1$:** $\frac{d^2y}{dx^2} = 12(-1) - 6 = -18$.
       - Since $-18 < 0$ (negative), this is a **Local Maximum**.

e. **Find Coordinates (using $y = 2x^3 - 3x^2 - 12x + 1$):**
    - **At $x = 2$: $y = 2(8) - 3(4) - 12(2) + 1 = 16 - 12 - 24 + 1 = -19$.**
    - **At $x = -1$: $y = 2(-1) - 3(1) - 12(-1) + 1 = -2 - 3 + 12 + 1 = 8$.**
  - **Answer:**
    - Local Maximum at **(-1, 8)**.
    - Local Minimum at **(2, -19)**.

---

3. APPLICATIONS (OPTIMIZATION PROBLEMS)

This is the process of finding the "best" (maximum or minimum) value in a real-world problem.

- **Worked Example 2:** A farmer has 100 meters of fencing to make a rectangular pen along a straight, existing wall. What is the maximum possible area of the pen?
  a. **Draw and Define:**
    - Let the width (perpendicular to the wall) be x.
    - The length (parallel to the wall) will be y.
    - The wall forms one side, so the fence needed is $2x + y$.
  b. **Form Constraint Equation:**
    - $2x + y = 100$.
    - Therefore, $y = 100 - 2x$.
  c. **Form Objective Equation (what to maximize):**
    - Area $A = x \cdot y$
  d. **Write A in terms of *one* variable (substitute y):**
    - $A(x) = x(100 - 2x) = 100x - 2x^2$
  e. **Find Maximum (using calculus):**
    - Find the derivative: $\frac{dA}{dx} = 100 - 4x$.
    - Set derivative to 0: $100 - 4x = 0 \Rightarrow 4x = 100 \Rightarrow x = 25$.
  f. **Verify it's a Maximum:**
    - Find the second derivative: $\frac{d^2A}{dx^2} = -4$.

- Since this is negative, it is a **Maximum**.

g. **Find the Area:**

- The dimensions are $x = 25$m and $y = 100 - 2(25) = 50$m.
- Max Area $= 25 \times 50 = 1250\ m^2$.

- **Answer:** The maximum possible area is **1250** m^2.

EVALUATION

1. What is a "stationary point"? What is the mathematical condition for finding one?
2. What is the "Second Derivative Test"? How does it distinguish a maximum from a minimum?
3. Find and classify the stationary points of the curve $y = x^3 - 12x + 1$.
4. If $dy/dx = 0$ at $x = 3$, and $d^2y/dx^2 = -5$ at $x = 3$, what kind of point is it?
5. Find the maximum value of the function $f(x) = 10 - 4x - x^2$.

ASSIGNMENT

1. Find the coordinates of the turning points of the curve $y = x^3 - 3x^2 + 1$ and determine the nature of each.
2. Find the minimum value of $y = 2x^2 + 8x - 5$.
3. The sum of two numbers is 12. Find the numbers if the product of one and the square of the other is a maximum. (Hint: $x + y = 12$, $P = x \cdot y^2$).
4. A closed cylindrical can is to be made to hold a volume of $1000\pi\ cm^3$. Find the dimensions (radius r and height h) that will *minimize* the amount of metal (surface area) used.
   - (Hint: $V = \pi r^2 h = 1000\pi$. $A = 2\pi r^2 + 2\pi rh$. Use V to write h in terms of r, substitute into A, and find $dA/dr = 0$).
5. Define: Stationary Point, Concave Up, Concave Down, Point of Inflexion, Optimization.

WEEK 9: THE CIRCLE

1. DEFINITION

A **circle** is the **locus** (set) of all points $P(x, y)$ that are **equidistant** from a fixed central point $C(h, k)$.

- The fixed point (h, k) is the **center**.
- The constant distance is the **radius** (r).

---

2. GENERAL EQUATION OF A CIRCLE

We can derive the equation using the distance formula. The distance between any point (x, y) on the circle and the center (h, k) must always be r.

- Distance formula: $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
- $r = \sqrt{(x - h)^2 + (y - k)^2}$
- Square both sides:

  $(x - h)^2 + (y - k)^2 = r^2$ This is the **standard equation (or center-radius form)** of a circle.

- **Special Case (Center at Origin):** If the center is $(0,0)$, the equation becomes: $x^2 + y^2 = r^2$
- **Worked Example 1:** Find the equation of a circle with center $(3, -5)$ and radius 4.
  - $h = 3, k = -5, r = 4.$
  - $(x - 3)^2 + (y - (-5))^2 = 4^2$
  - $(x - 3)^2 + (y + 5)^2 = 16.$

---

3. GENERAL FORM ($x^2 + y^2 + 2gx + 2fy + c = 0$)

If we *expand* the standard form, we get the general form. $(x - h)^2 + (y - k)^2 = r^2$ $(x^2 - 2hx + h^2) + (y^2 - 2ky + k^2) = r^2$ $x^2 + y^2 - 2hx - 2ky + (h^2 + k^2 - r^2) = 0$

We can compare this to the form $x^2 + y^2 + 2gx + 2fy + c = 0$. By comparing coefficients:

- $2g = -2h \Rightarrow \mathbf{h = -g}$
- $2f = -2k \Rightarrow \mathbf{k = -f}$
- $c = h^2 + k^2 - r^2 \Rightarrow r^2 = h^2 + k^2 - c \Rightarrow r^2 = (-g)^2 + (-f)^2 - c$

From the General Form, we can find the center and radius:

- **Center** $(h, k) = (-g, -f)$
- **Radius** $r = \sqrt{g^2 + f^2 - c}$
- **Worked Example 2:** Find the center and radius of the circle $x^2 + y^2 - 6x + 10y + 9 = 0$.
    - **Method A (Comparing Coefficients):**
        i. Compare to $x^2 + y^2 + 2gx + 2fy + c = 0$.
        ii. $2g = -6 \Rightarrow \mathbf{g = -3}$
        iii. $2f = 10 \Rightarrow \mathbf{f = 5}$
        iv. $c = 9$
        v. **Center** $= (-g, -f) = (3, -5)$
        vi. **Radius** $= \sqrt{g^2 + f^2 - c} = \sqrt{(-3)^2 + 5^2 - 9} = \sqrt{9 + 25 - 9} = \sqrt{25} = 5$.
    - **Method B (Completing the Square):** (Often safer)
        vii. Group x and y terms: $(x^2 - 6x) + (y^2 + 10y) + 9 = 0$
        viii. Complete the square for x: $(x^2 - 6x + 9) \Rightarrow (x - 3)^2$
        ix. Complete the square for y: $(y^2 + 10y + 25) \Rightarrow (y + 5)^2$

x. Add the new numbers to *both* sides: $(x^2 - 6x + 9) + (y^2 + 10y + 25) = -9 + 9 + 25$

xi. Factor: $(x - 3)^2 + (y + 5)^2 = 25$

xii. Compare to $(x - h)^2 + (y - k)^2 = r^2$:

- $h = 3, k = -5 \Rightarrow$ **Center = (3, -5)**
- $r^2 = 25 \Rightarrow$ **Radius = 5**

---

4. EQUATION OF TANGENT TO A CIRCLE

A tangent is a line that touches the circle at *one point*. The tangent is *always* **perpendicular** to the radius at the point of contact.

Steps to find the equation of the tangent:

1. Find the **Center** (h, k) of the circle.
2. Find the **Gradient of the Radius** (m_{rad}) from the center (h, k) to the point of contact (x_1, y_1). $m_{rad} = \frac{y_1 - k}{x_1 - h}$
3. Find the **Gradient of the Tangent** (m_{tan}). It is the negative reciprocal of the radius gradient. $m_{tan} = -\frac{1}{m_{rad}}$
4. Use the line equation $y - y_1 = m_{tan}(x - x_1)$.

- **Worked Example 3:** Find the equation of the tangent to the circle $(x - 1)^2 + (y - 2)^2 = 10$ at the point $(2,5)$.
    a. **Center:** $(h, k) = (1,2)$.
    b. **Point:** $(x_1, y_1) = (2,5)$.
    c. **Find m_{rad}:** $m_{rad} = \frac{5-2}{2-1} = \frac{3}{1} = 3$.
    d. **Find m_{tan}:** $m_{tan} = -\frac{1}{3}$.
    e. **Find Equation:**
        - $y - 5 = -\frac{1}{3}(x - 2)$
        - $3(y - 5) = -1(x - 2)$

- $3y - 15 = -x + 2$
- $x + 3y - 17 = 0$

5. LENGTH OF TANGENT

The length L of a tangent from an external point $P(x_1, y_1)$ to a circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is given by: $L^2 = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$ (You just plug the point into the circle's equation).

- **Worked Example 4:** Find the length of the tangent from the point $(5,6)$ to the circle $x^2 + y^2 - 4x + 2y - 10 = 0$.
  f. $L^2 = (5)^2 + (6)^2 - 4(5) + 2(6) - 10$
  g. $L^2 = 25 + 36 - 20 + 12 - 10$
  h. $L^2 = 73 - 30 = 43$
  i. $L = \sqrt{43}$ units.

EVALUATION

1. Find the equation of a circle with center $(-2,3)$ and radius 7.
2. Find the center and radius of the circle $x^2 + y^2 + 8x - 4y + 11 = 0$.
3. Find the equation of the tangent to the circle $x^2 + y^2 = 25$ at the point $(3,4)$.
4. Find the length of the tangent from the point $(1,1)$ to the circle $x^2 + y^2 - 6x - 8y + 2 = 0$.
5. Find the equation of the circle that has a diameter with endpoints $A(1,2)$ and $B(7,10)$. (Hint: The center is the *midpoint* of AB. The radius is *half* the distance AB).

ASSIGNMENT

1. Find the center and radius of the circle $2x^2 + 2y^2 - 12x + 8y - 1 = 0$. (Hint: You must divide by 2 *first*).
2. Find the equation of the tangent and the normal to the circle $x^2 + y^2 - 10x + 2y + 1 = 0$ at the point $(1, -4)$.

3. Find the length of the tangent from the origin $(0,0)$ to the circle $x^2 + y^2 - 12x - 10y + 15 = 0$.
4. A circle with center $(3, -2)$ passes through the point $(6,2)$. Find the equation of the circle. (Hint: Find the radius r first, using the distance formula).
5. Show that the circle $x^2 + y^2 - 2x - 4y = 0$ passes through the origin. Find the equation of the tangent at the origin.

WEEK 10: CONIC SECTIONS I

1. DEFINITION OF CONIC SECTIONS

A **conic section** (or "conic") is a curve obtained by intersecting a plane with a **double-napped cone**.

1. **Circle:** The plane cuts *horizontally* (parallel to the base).
2. **Ellipse:** The plane cuts at a *shallow angle*.
3. **Parabola:** The plane cuts *parallel to the side* of the cone.
4. **Hyperbola:** The plane cuts *steeply*, intersecting *both* nappes of the cone.

2. ALTERNATIVE DEFINITION (FOCUS-DIRECTRIX)

A conic section is the **locus** (set) of a point P that moves in a plane such that its distance from a fixed point (the **Focus**, F) is in a constant ratio e (the **eccentricity**) to its perpendicular distance from a fixed line (the **Directrix**, L).

$$PF = e \cdot PL$$

This ratio e defines the shape of the conic:

- $e = 0$: **Circle** (a special case of an ellipse)
- $0 < e < 1$: **Ellipse**
- $e = 1$: **Parabola**
- $e > 1$: **Hyperbola**

---

3. THE PARABOLA ($e = 1$)

- **Definition:** The set of all points $P(x, y)$ that are equidistant from a focus F and a directrix L. $PF = PL$.
- **Standard Equation (Vertex at origin):**
  - $y^2 = 4ax$

- **Properties:**
    - Opens *Right* (if $a > 0$) or *Left* (if $a < 0$).
    - **Vertex:** $(0,0)$
    - **Focus:** $(a, 0)$
    - **Directrix:** $x = -a$
- **Other Standard Equation (Vertex at origin):**
    - $x^2 = 4ay$
- **Properties:**
    - Opens *Up* (if $a > 0$) or *Down* (if $a < 0$).
    - **Vertex:** $(0,0)$
    - **Focus:** $(0, a)$
    - **Directrix:** $y = -a$
- **Worked Example 1:** Find the focus and directrix of the parabola $y^2 = 12x$.
    a. **Compare:** $y^2 = 4ax$ with $y^2 = 12x$.
    b. **Find a:** $4a = 12 \Rightarrow a = 3$.
    c. **Find properties:**
        - **Vertex:** $(0, 0)$
        - **Focus:** $(a, 0) = \mathbf{(3, 0)}$
        - **Directrix:** $x = -a \Rightarrow \mathbf{x = -3}$

---

4. THE ELLIPSE ($0 < e < 1$)

- **Definition:** The set of all points P such that the *sum* of the distances from P to two fixed points (the **foci**, F_1 and F_2) is constant.
- **Standard Equation (Center at origin):**
    - $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

- **Properties (assuming $a > b$):**
    - **Major Axis:** Horizontal, length $2a$. **Vertices:** $(\pm a, 0)$.
    - **Minor Axis:** Vertical, length $2b$. **Co-vertices:** $(0, \pm b)$.
    - **Foci (F_1, F_2):** $(\pm c, 0)$, where $\mathbf{c^2 = a^2 - b^2}$.
    - **Eccentricity:** $\mathbf{e = c/a}$.
- **Worked Example 2:** Find the vertices, foci, and eccentricity of $\frac{x^2}{25} + \frac{y^2}{16} = 1$.
    a. **Identify a^2 and b^2:** $a^2 = 25 \Rightarrow a = 5$. $b^2 = 16 \Rightarrow b = 4$.
    b. **Find c:** $c^2 = a^2 - b^2 = 25 - 16 = 9 \Rightarrow c = 3$.
    c. **Find properties:**
        - **Vertices:** $(\pm a, 0) = (\pm \mathbf{5}, \mathbf{0})$
        - **Co-vertices:** $(0, \pm b) = (\mathbf{0}, \pm \mathbf{4})$
        - **Foci:** $(\pm c, 0) = (\pm \mathbf{3}, \mathbf{0})$
        - **Eccentricity:** $e = c/a = \mathbf{3/5}$. (Note: $3/5 < 1$, as expected).

---

5. THE HYPERBOLA ($e > 1$)

- **Definition:** The set of all points P such that the *difference* of the distances from P to two fixed points (the **foci**, F_1 and F_2) is constant.
- **Standard Equation (Center at origin, opens L/R):**
    - $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$
- **Properties:**
    - **Transverse Axis:** Horizontal. **Vertices:** $(\pm a, 0)$.
    - **Foci (F_1, F_2):** $(\pm c, 0)$, where $\mathbf{c^2 = a^2 + b^2}$.
    - **Eccentricity:** $\mathbf{e = c/a}$.
    - **Asymptotes:** The "X" shape the graph approaches. $y = \pm \frac{b}{a}x$.
- **Worked Example 3:** Find the vertices, foci, eccentricity, and asymptotes for $\frac{x^2}{9} - \frac{y^2}{16} = 1$.

a. **Identify a^2 and b^2:** $a^2 = 9 \Rightarrow a = 3$. $b^2 = 16 \Rightarrow b = 4$.

b. **Find c:** $c^2 = a^2 + b^2 = 9 + 16 = 25 \Rightarrow c = 5$.

c. **Find properties:**

- **Vertices:** $(\pm a, 0) = (\pm \mathbf{3}, \mathbf{0})$
- **Foci:** $(\pm c, 0) = (\pm \mathbf{5}, \mathbf{0})$
- **Eccentricity:** $e = c/a = \mathbf{5/3}$. (Note: $5/3 > 1$, as expected).
- **Asymptotes:** $y = \pm \frac{b}{a}x = \pm \frac{\mathbf{4}}{\mathbf{3}}\mathbf{x}$.

EVALUATION

1. How are the four conic sections formed from a cone?
2. What is the "focus-directrix" definition of a conic section?
3. What value of eccentricity e corresponds to (a) a parabola, (b) an ellipse, (c) a hyperbola?
4. Find the focus and directrix for the parabola $x^2 = 16y$.
5. Find the vertices and foci for the ellipse $\frac{x^2}{169} + \frac{y^2}{144} = 1$.

ASSIGNMENT

1. Find the focus and directrix of the parabola $y^2 = -8x$.
2. Find the vertices, foci, and asymptotes for the hyperbola $\frac{x^2}{64} - \frac{y^2}{36} = 1$.
3. Find the equation of a parabola with its vertex at the origin and its focus at $(0, -5)$.
4. Find the equation of an ellipse with its center at the origin, vertices at $(\pm 10,0)$, and foci at $(\pm 8,0)$.
5. What is the "latus rectum" of a parabola? (Research this). Find its length for $y^2 = 4ax$.

WEEK 11: CONIC SECTIONS II

1. TOPIC: PROPERTIES AND GRAPHS OF CONIC SECTIONS

This week, we use the properties we learned in Week 10 to accurately **sketch the graphs** of each conic section.

2. GRAPHING THE PARABOLA

- **Equation:** $y^2 = 4ax$ (opens Right/Left) or $x^2 = 4ay$ (opens Up/Down).
- **Key Info Needed:**
    a. **Vertex:** The "turning point" of the curve.
    b. **Focus:** The point "inside" the curve.
    c. **Directrix:** The line "behind" the curve.
    d. **Latus Rectum:** A chord (line segment) that passes through the *focus* and is *perpendicular* to the axis of symmetry. Its length is $|\mathbf{4a}|$. This tells us **how "wide"** the parabola is at its focus.
- **Worked Example 1:** Sketch the graph of the parabola $x^2 = -8y$.
    a. **Identify Type:** It is of the form $x^2 = 4ay$. Since $4a$ is negative, it opens **downwards**.
    b. **Find a:** $4a = -8 \Rightarrow a = -2$.
    c. **Find Properties:**
        - **Vertex:** $(0,0)$.
        - **Focus:** $(0, a) = (\mathbf{0}, \mathbf{-2})$.
        - **Directrix:** $y = -a \Rightarrow \mathbf{y = 2}$.
        - **Latus Rectum:** Length = $|4a| = |-8| = 8$. This means the parabola is 8 units wide at the focus (4 units left, 4 units right).
    d. **Sketch Graph:**
        - Plot the vertex at (0, 0).

- Plot the focus at (0, -2).
- Draw the directrix (a dashed line) at $y = 2$.
- Draw the Latus Rectum points (4 units left/right of the focus): $(-4, -2)$ and $(4, -2)$.
- Draw a smooth U-shaped curve passing through the vertex and the Latus Rectum points.

---

3. GRAPHING THE ELLIPSE

- **Equation:** $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$
- **Key Info Needed:**
  a. **Center:** (0,0).
  b. **Vertices (Major Axis):** Find the *larger* of a^2 or b^2.
  c. **Co-vertices (Minor Axis):** The *smaller* value.
  d. **Foci:** $c^2 = a^2 - b^2$.
- **Worked Example 2:** Sketch the graph of $\frac{x^2}{9} + \frac{y^2}{25} = 1$.
  a. **Identify a^2, b^2:** $a^2 = 9 \Rightarrow a = 3$ (x-direction). $b^2 = 25 \Rightarrow b = 5$ (y-direction).
  b. **Identify Axes:** Since $b > a$, the **Major Axis is Vertical**.
  c. **Find Properties:**
     - **Center:** (0,0).
     - **Vertices (Major):** $(0, \pm b) = (\mathbf{0}, \pm \mathbf{5})$.
     - **Co-vertices (Minor):** $(\pm a, 0) = (\pm \mathbf{3}, \mathbf{0})$.
  d. **Sketch Graph:**
     - Plot the center (0, 0).
     - Plot the 4 points: (0, 5), (0, -5), (3, 0), (-3, 0).
     - Draw a smooth, oval shape connecting these four points.

---

4. GRAPHING THE HYPERBOLA

- **Equation:** $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ (opens L/R) or $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$ (opens U/D).
- **Key Info Needed:**
    a. **Center:** (0,0).
    b. **Vertices:** On the "positive" axis.
    c. **Asymptotes:** The (dashed) "X" that guides the curve.
- **Worked Example 3:** Sketch the graph of $\frac{x^2}{16} - \frac{y^2}{9} = 1$.
    a. **Identify Type:** The x^2 term is positive, so it opens **Left/Right**.
    b. **Find a, b:** $a^2 = 16 \Rightarrow a = 4$. $b^2 = 9 \Rightarrow b = 3$.
    c. **Find Properties:**
        - **Center:** (0,0).
        - **Vertices (Transverse):** $(\pm a, 0) = (\pm \mathbf{4}, \mathbf{0})$.
    d. **Find Asymptotes:** $y = \pm \frac{b}{a}x = \pm \frac{3}{4}x$.
    e. **Sketch Graph (using the "Box Method"):**
        - Draw a "central box" that goes to $\pm a$ on the x-axis and $\pm b$ on the y-axis (i.e., from -4 to 4 on x, -3 to 3 on y).
        - Draw the asymptotes (dashed lines) as the diagonals *through* this box.
        - Plot the vertices at (4, 0) and (-4, 0).
        - Draw the hyperbola curves starting from the vertices and *approaching* (but never touching) the asymptotes.

EVALUATION

1. What does the "Latus Rectum" tell you about a parabola's graph?
2. Sketch the graph of the parabola $y^2 = 4x$. Label the vertex, focus, and directrix.
3. Sketch the graph of the ellipse $\frac{x^2}{100} + \frac{y^2}{64} = 1$. Label the vertices and co-vertices.
4. What is the "central box" used for in graphing a hyperbola?

5. Sketch the graph of the hyperbola $\frac{y^2}{4} - \frac{x^2}{1} = 1$. (Hint: This opens Up/Down).

ASSIGNMENT

1. Find the properties (vertex, focus, directrix) of the parabola $x^2 = -12y$ and sketch its graph.
2. Find the properties (vertices, co-vertices, foci) of the ellipse $\frac{x^2}{16} + \frac{y^2}{49} = 1$ and sketch its graph.
3. Find the properties (vertices, foci, asymptotes) of the hyperbola $\frac{x^2}{25} - \frac{y^2}{25} = 1$ and sketch its graph.
4. An ellipse has its center at $(0,0)$, a focus at $(12,0)$, and a vertex at $(13,0)$. Find its equation.
5. A parabola has its vertex at $(0,0)$ and its directrix at $x = 5$. What is its equation?

PAWILLS CONSULTING

WEEK 12: CONIC SECTIONS III

1. TOPIC: SHIFTED CONICS AND THE GENERAL EQUATION

So far, all our conics have been centered at the origin $(0,0)$. This week, we learn how to handle conics that have been **translated (shifted)** to a new center (h, k).

2. STANDARD EQUATIONS FOR SHIFTED CONICS

The formula is a simple translation:

- Replace x with $(x - h)$
- Replace y with $(y - k)$

Conic	Center at (0, 0)	Center at (h, k)
Circle	$x^2 + y^2 = r^2$	$(x-h)^2 + (y-k)^2 = r^2$
Parabola (opens R/L)	$y^2 = 4ax$	$(y-k)^2 = 4a(x-h)$
Parabola (opens U/D)	$x^2 = 4ay$	$(x-h)^2 = 4a(y-k)$
Ellipse (horizontal)	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$
Hyperbola (horizontal)	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$

- **Properties:** All properties are just *shifted* from the origin.
    - **Ellipse:** Center is (h, k), Vertices are $(h \pm a, k)$.
    - **Parabola:** Vertex is (h, k), Focus is $(h + a, k)$.
- **Worked Example 1:** Find the equation of an ellipse with center $(5, -1)$, major axis horizontal with length 10, and minor axis length 8.
    a. **Center:** $(h, k) = (5, -1)$.
    b. **Axes:** Major axis $2a = 10 \Rightarrow a = 5$. Minor axis $2b = 8 \Rightarrow b = 4$.

c. **Equation:** $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$

- $\frac{(x-5)^2}{5^2} + \frac{(y-(-1))^2}{4^2} = 1$
- $\frac{(x-5)^2}{25} + \frac{(y+1)^2}{16} = 1$

---

3. THE GENERAL FORM: $Ax^2 + Cy^2 + Dx + Ey + F = 0$

This is the "unsimplified" form of a conic. Our job is to use the **Completing the Square** method to force it back into a standard (shifted) form, so we can identify it.

How to Identify (without completing the square):

- If $A = C$ (and $A, C \neq 0$): It's a **Circle**.
- If A and C have the *same sign* but $A \neq C$: It's an **Ellipse**.
- If A and C have *opposite signs*: It's a **Hyperbola**.
- If *either* $A = 0$ or $C = 0$ (but not both): It's a **Parabola**.
- **Worked Example 2:** Identify the conic $9x^2 + 16y^2 - 18x + 64y - 71 = 0$. Find its center and properties.

  a. **Identify:** $A = 9, C = 16$. Both are positive and $A \neq C$. It is an **Ellipse**.

  b. **Complete the Square:**

  - Group x and y terms: $(9x^2 - 18x) + (16y^2 + 64y) = 71$
  - Factor out coefficients: $9(x^2 - 2x) + 16(y^2 + 4y) = 71$
  - **Complete for x:** Half of -2 is -1. $(-1)^2 = 1$. Add 1 *inside* the x-bracket.
  - **Complete for y:** Half of 4 is 2. $(2)^2 = 4$. Add 4 *inside* the y-bracket.
  - **Balance the Equation:** We didn't just add 1 and 4. We added $9 \times 1 = 9$ and $16 \times 4 = 64$.
    - $9(x^2 - 2x + 1) + 16(y^2 + 4y + 4) = 71 + 9 + 64$
  - **Factor:** $9(x-1)^2 + 16(y+2)^2 = 144$

- **Set to 1:** Divide the *entire* equation by 144.
  - $\frac{9(x-1)^2}{144} + \frac{16(y+2)^2}{144} = \frac{144}{144}$
- **Final Standard Form:** $\frac{(x-1)^2}{16} + \frac{(y+2)^2}{9} = 1$

c. **Find Properties:**

- **Center (h, k):** $(1, -2)$
- $a^2 = 16 \Rightarrow a = 4$. Major axis is horizontal.
- $b^2 = 9 \Rightarrow b = 3$.
- **Vertices:** $(h \pm a, k) \Rightarrow (1 \pm 4, -2) \Rightarrow (5, -2)$ and $(-3, -2)$.

- **Worked Example 3:** Identify the conic $y^2 - 8x + 6y + 17 = 0$.

a. **Identify:** $A = 0$ (no x^2 term), $C = 1$. It is a **Parabola**.

b. **Complete the Square:**

- Group y terms. Move x and F to the other side.
- $(y^2 + 6y) = 8x - 17$
- **Complete for y:** Half of 6 is 3. $(3)^2 = 9$. Add 9 to *both* sides.
- $(y^2 + 6y + 9) = 8x - 17 + 9$
- **Factor:** $(y + 3)^2 = 8x - 8$
- Factor out the coefficient of x: $(y + 3)^2 = 8(x - 1)$

c. **Find Properties:**

- **Compare to:** $(y - k)^2 = 4a(x - h)$
- **Vertex (h, k):** $(1, -3)$
- **Find a:** $4a = 8 \Rightarrow a = 2$.
- **Focus:** $(h + a, k) = (1 + 2, -3) = (3, -3)$.

EVALUATION

1. Write the standard equation for a parabola with vertex $(2,5)$ that opens upwards.
2. Write the standard equation for an ellipse with center $(-1,0)$, horizontal major axis of 10, and minor axis of 6.

3. Identify the conic given by the equation $x^2 - y^2 - 4x + 8y - 21 = 0$ (based on A and C).
4. Find the center of the circle $x^2 + y^2 + 10x - 6y + 30 = 0$ by completing the square.
5. Find the vertex of the parabola $x^2 + 4x - 4y = 0$ by completing the square.

ASSIGNMENT

1. Find the center and vertices of the ellipse $25x^2 + 4y^2 - 100x - 40y + 100 = 0$.
2. Find the vertex, focus, and directrix of the parabola $y^2 + 8y - 6x + 4 = 0$.
3. Find the center, vertices, and asymptotes of the hyperbola $9x^2 - 16y^2 - 18x - 64y - 199 = 0$.
4. Find the equation of the ellipse with vertices at $(4,3)$ and $(4,13)$ and co-vertices at $(1,8)$ and $(7,8)$.
5. What conic is represented by $3x^2 - 2y^2 - 6x - 12y - 27 = 0$? Justify your answer.

PAWILLS CONSULTING

WEEK 13: SAMPLE SPACE AND EVENTS

1. PROBABILITY: BASIC CONCEPTS

- **Probability** is the measure of the *likelihood* that an event will occur. It is a number between 0 and 1.
    - $P(E) = 0$ means the event is impossible.
    - $P(E) = 1$ means the event is certain.
- **Experiment:** An action with an uncertain outcome (e.g., rolling a die).
- **Sample Space (S):** The set of *all possible outcomes* of an experiment.
- **Event (E):** A *subset* of the sample space; a specific, "favorable" outcome.
- **Formula:** $P(E) = \frac{\text{Number of favorable outcomes, } n(E)}{\text{Total number of possible outcomes, } n(S)}$

---

2. SAMPLE SPACE AND EVENTS

Listing the sample space is the most important first step.

- **Worked Example 1: Tossing one coin**
    - $S = \{\text{Head, Tail}\}$
    - $n(S) = 2$
- **Worked Example 2: Rolling one die**
    - $S = \{1,2,3,4,5,6\}$
    - $n(S) = 6$
    - Event E = "getting a prime number"
    - $E = \{2,3,5\}$
    - $n(E) = 3$
    - $P(E) = n(E)/n(S) = 3/6 = 1/2$.
- **Worked Example 3: Tossing two coins**
    - $S = \{\text{HH, HT, TH, TT}\}$ (Note: HT and TH are different)

- $n(S) = 4$
- Event F = "getting at least one head"
- $F = \{HH, HT, TH\}$
- $n(F) = 3$
- $P(F) = 3/4$.

- **Worked Example 4: Rolling two dice**
  - $n(S) = 6 \times 6 = 36$ outcomes. (This is best shown in a 6x6 table).
  - Event G = "getting a total sum of 5"
  - $G = \{(1,4), (2,3), (3,2), (4,1)\}$
  - $n(G) = 4$
  - $P(G) = 4/36 = 1/9$.

---

3. INDEPENDENT AND DEPENDENT EVENTS

3A. INDEPENDENT EVENTS

- **Definition:** Two events are **independent** if the outcome of the first event does *not* affect the probability of the second event.
- **Key word:** "With replacement."
- **Rule:** $P(A \text{ and } B) = P(A) \times P(B)$
- **Worked Example 5:** A bag contains 4 red balls and 6 blue balls. A ball is drawn, its color is noted, and it is **replaced**. A second ball is drawn. What is the probability of drawing two red balls?

  a. $P(\text{1st Red}) = 4/(4 + 6) = 4/10$.

  b. Since the ball is *replaced*, the bag is unchanged.

  c. $P(\text{2nd Red}) = 4/10$.

  d. $P(\text{Red and Red}) = P(\text{1st Red}) \times P(\text{2nd Red}) = \frac{4}{10} \times \frac{4}{10} = \frac{16}{100} = \frac{4}{25}$.

3B. DEPENDENT EVENTS

- **Definition:** Two events are **dependent** if the outcome of the first event *changes* the probability of the second event.
- **Key word:** "Without replacement."
- **Rule:** $P(A \text{ and } B) = P(A) \times P(B|A)$
    - ($P(B|A)$ means "the probability of B, *given that* A has already happened").
- **Worked Example 6:** A bag contains 4 red balls and 6 blue balls. A ball is drawn, set aside, and **not replaced**. A second ball is drawn. What is the probability of drawing two red balls?
    a. $P(\text{1st Red}) = 4/(4+6) = 4/10$.
    b. **Now, one red ball is gone.** The bag now contains 3 red and 6 blue (only 9 balls total).
    c. $P(\text{2nd Red}|\text{1st was Red}) = 3/9$.
    d. $P(\text{Red and Red}) = P(\text{1st Red}) \times P(\text{2nd Red}|\text{1st Red})$
        - $= \frac{4}{10} \times \frac{3}{9} = \frac{12}{90} = \frac{2}{15}$.

EVALUATION

1. What is a "sample space"? What is the sample space for tossing 3 coins?
2. What is the probability of rolling a single die and getting a number greater than 4?
3. What is the main difference between an independent and a dependent event?
4. Two dice are rolled. What is the probability of getting a "double" (e.g., (1,1), (2,2), etc.)?
5. A box contains 5 good mangoes and 3 bad mangoes. If two mangoes are picked one after the other *without replacement*, what is the probability that *both* are good?

ASSIGNMENT

1. A standard deck of 52 playing cards is used.
    - (a) What is $P(\text{drawing a King})$?

- (b) What is $P(\text{drawing a Heart})$?
- (c) What is $P(\text{drawing a red card or a 7})$? (Hint: Use $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$).

2. A bag contains 5 red, 3 green, and 2 yellow balls. A ball is drawn and *replaced*. A second ball is drawn. Find the probability of getting:
   - (a) Two green balls.
   - (b) A red ball, then a yellow ball.
3. Repeat question 2, but this time the first ball is **NOT** replaced.
4. Draw a tree diagram for the experiment of tossing 3 coins. Use it to find the probability of getting *exactly* two heads.
5. Define: Experiment, Outcome, Event, Sample Space, Mutually Exclusive.

PAWILLS CONSULTING

SS2 FURTHER MATHEMATICS LESSON NOTES

THIRD TERM

WEEK 1: SAMPLE SPACE AND EVENTS (CONTINUED)

1. RECAP: BASIC PROBABILITY

In SS1, we defined probability as the measure of the likelihood of an event.

- **Experiment:** An action with an uncertain outcome (e.g., rolling a die).
- **Sample Space (S):** The set of *all possible outcomes*. $n(S)$ is the total number of outcomes.
- **Event (E):** A *subset* of the sample space (a "favorable" outcome). $n(E)$ is the number of favorable outcomes.
- **Probability:** $P(E) = \frac{n(E)}{n(S)}$, where $0 \leq P(E) \leq 1$.
- **Complement:** $P(E') = P(\text{not } E) = 1 - P(E)$.

---

2. PROBABILITY OF COMPOUND EVENTS

Compound events involve two or more simple events. We use **set theory** to understand them. The "or" connective (∪, Union) and the "and" connective (∩, Intersection) are key.

2A. THE ADDITION LAW (FOR "OR" EVENTS)

The Addition Law is used to find the probability of event A *or* event B happening. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ This formula prevents "double counting" the outcomes that are in *both* A and B.

2B. MUTUALLY EXCLUSIVE EVENTS

- **Definition:** Two events, A and B, are **mutually exclusive** if they **cannot happen at the same time**. They have no outcomes in common.
- In set terms: $A \cap B = \emptyset$ (the empty set).

- Therefore: $P(A \cap B) = 0$.
- **Simplified Addition Law (for M.E. Events):** $P(A \cup B) = P(A) + P(B)$
- **Worked Example 1:** A single die is rolled. What is the probability of rolling a 2 *or* a 5?
    a. **Event A (rolling a 2):** $P(A) = 1/6$
    b. **Event B (rolling a 5):** $P(B) = 1/6$
    c. **Check:** Can you roll a 2 and a 5 at the same time? No. The events are **mutually exclusive**.
    d. **Solve:** $P(A \cup B) = P(A) + P(B) = 1/6 + 1/6 = 2/6 = 1/3$.
- **Worked Example 2:** A card is drawn from a standard deck of 52. What is the probability of drawing a King *or* a Heart?
    a. **Event A (King):** $P(A) = 4/52$
    b. **Event B (Heart):** $P(B) = 13/52$
    c. **Check:** Can an event be both? Yes, the "King of Hearts".
    d. **Intersection:** $P(A \cap B) = P(\text{King of Hearts}) = 1/52$.
    e. **Solve (General Rule):** $(A \cup B) = P(A) + P(B) - P(A \cap B)$ $P(A \cup B) = \frac{4}{52} + \frac{13}{52} - \frac{1}{52} = \frac{16}{52} = \frac{4}{13}$.

2C. THE MULTIPLICATION LAW (FOR "AND" EVENTS)

The Multiplication Law is used to find the probability of event A *and* event B happening *in sequence*. $P(A \cap B) = P(A) \times P(B|A)$ where $P(B|A)$ is the "conditional probability" of B happening, *given that* A has already happened.

2D. INDEPENDENT EVENTS

- **Definition:** Two events, A and B, are **independent** if the outcome of event A *does not* affect the probability of event B.
- This means $P(B|A) = P(B)$.

- This typically occurs in problems **"with replacement."**
- **Simplified Multiplication Law (for Independent Events):** $P(A \cap B) = P(A) \times P(B)$
- **Worked Example 3:** A bag contains 5 red balls and 3 blue balls. A ball is drawn, its color noted, and it is **replaced**. A second ball is drawn. What is the probability of drawing a Red ball *and* then a Blue ball?
    a. **Event A (1st is Red):** $P(A) = 5/8$.
    b. **Event B (2nd is Blue):** Since the first ball was *replaced*, the bag is unchanged. The events are **independent**.
    c. $P(B) = 3/8$.
    d. **Solve:** $P(A \cap B) = P(A) \times P(B) = \frac{5}{8} \times \frac{3}{8} = \frac{15}{64}$.

2E. DEPENDENT EVENTS

- **Definition:** Two events are **dependent** if the outcome of event A *changes* the probability of event B.
- This typically occurs in problems **"without replacement."**
- **Multiplication Law (for Dependent Events):** $P(A \cap B) = P(A) \times P(B|A)$
- **Worked Example 4:** A bag contains 5 red balls and 3 blue balls. A ball is drawn and **not replaced**. A second ball is drawn. What is the probability of drawing a Red ball *and* then a Blue ball?
    a. **Event A (1st is Red):** $P(A) = 5/8$.
    b. **Event B (2nd is Blue):** The first ball (Red) is gone. The bag now contains 4 red and 3 blue balls, for a *new total* of 7 balls.
    c. The probability of B *given that* A happened is $P(B|A) = 3/7$.
    d. **Solve:** $P(A \cap B) = P(A) \times P(B|A) = \frac{5}{8} \times \frac{3}{7} = \frac{15}{56}$.

EVALUATION

1. What is the key difference between Mutually Exclusive events and Independent events?
2. A card is drawn from a deck. What is the probability it is a 7 *or* a Spade?
3. A die is rolled and a coin is tossed. What is the probability of getting a "Head" *and* a "6"?
4. A box contains 10 spanners. 3 are defective. If two spanners are picked one after the other *without replacement*, what is the probability that *both* are defective?
5. If $P(A) = 0.5$ and $P(B) = 0.4$, and the events are independent, find $P(A \cup B)$. (Hint: First find $P(A \cap B)$).

ASSIGNMENT

1. A bag contains 6 red, 4 white, and 5 blue marbles. If 3 marbles are drawn *without replacement*, what is the probability of picking:
    - (a) Three red marbles?
    - (b) A red, then a white, then a blue marble (in that order)?
    - (c) All three of the same color?
2. In a class of 30 students, 15 study Physics, 12 study Chemistry, and 5 study *both*. What is the probability that a randomly selected student studies *neither* Physics *nor* Chemistry? (Hint: Draw a Venn diagram).
3. Explain, using an example, what "Conditional Probability" $P(B|A)$ means.
4. If $P(X) = 1/3$ and $P(Y) = 1/4$, and events X and Y are mutually exclusive, find $P(X \cup Y)$ and $P(X \cap Y)$.
5. If $P(M) = 1/3$ and $P(N) = 1/4$, and events M and N are independent, find $P(M \cup N)$ and $P(M \cap N)$.

WEEK 2: PERMUTATION AND COMBINATION I

1. FUNDAMENTAL COUNTING PRINCIPLE

This is the foundation for all permutations and combinations.

- **The Principle:** If one task (Event A) can be performed in m different ways, and a second independent task (Event B) can be performed in n different ways, then both tasks can be performed in $m \times n$ different ways.
- **Worked Example 1:** A man has 5 shirts and 3 pairs of trousers. How many different shirt-trouser outfits can he wear?
    - Task A (Choose shirt): $m = 5$ ways.
    - Task B (Choose trousers): $n = 3$ ways.
    - Total outfits = $5 \times 3 = 15$ ways.
- **Worked Example 2:** How many 3-digit numbers can be formed from the digits 1, 2, 3, 4, 5 if (a) repetition is allowed, and (b) repetition is not allowed?
    - (a) **Repetition Allowed:**
        - Choice for 1st digit: 5 ways
        - Choice for 2nd digit: 5 ways
        - Choice for 3rd digit: 5 ways
        - Total = $5 \times 5 \times 5 = 125$ numbers.
    - (b) **Repetition Not Allowed:**
        - Choice for 1st digit: 5 ways (e.g., you pick '3')
        - Choice for 2nd digit: 4 ways left (e.g., '1, 2, 4, 5')
        - Choice for 3rd digit: 3 ways left
        - Total = $5 \times 4 \times 3 = 60$ numbers.

2. FACTORIAL NOTATION

- **Definition:** $n!$ (read as "n-factorial") is the product of all positive integers from n down to 1.
- $n! = n \times (n-1) \times (n-2) \times \ldots \times 3 \times 2 \times 1$
- **Examples:**
  - $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$
  - $3! = 3 \times 2 \times 1 = 6$
  - $1! = 1$
- **Special Case (Definition):** $\mathbf{0! = 1}$. This is a mathematical convention necessary for the formulas to work.

---

3. PERMUTATIONS

- **Definition:** A **permutation** is an **arrangement** or **ordering** of a set of distinct objects.
- **Key Concept:** In permutations, **ORDER MATTERS**.
- **Example:** The arrangement "ABC" is a *different permutation* from "BAC".

3A. PERMUTATION OF n DISTINCT OBJECTS

- The number of ways to arrange n different objects in a line is $n!$.
- **Worked Example 3:** In how many ways can the letters of the word "MATHS" be arranged?
  - There are 5 distinct letters.
  - Number of arrangements = $5! = 120$ ways.

3B. PERMUTATION OF r OBJECTS FROM n

- This is the number of ways to *arrange* r objects that are *chosen* from a set of n objects.
- **Formula:** $P(n, r)$ or ^nP_r $^nP_r = \frac{n!}{(n-r)!}$
- **Worked Example 4:** In a race with 10 athletes, how many ways can the Gold, Silver, and Bronze medals be awarded?

- **Analyze:** Does order matter? Yes. (A-Gold, B-Silver) is different from (B-Gold, A-Silver). So, it is a permutation.
- We are *arranging* 3 people ($r = 3$) chosen from 10 people ($n = 10$).
- ${}^{10}P_3 = \frac{10!}{(10-3)!} = \frac{10!}{7!}$
- $= \frac{10\times9\times8\times7!}{7!} = 10 \times 9 \times 8 = 720$ ways.
- **(Counting Principle Method):** Gold: 10 choices. Silver: 9 choices. Bronze: 8 choices. Total = $10 \times 9 \times 8 = 720$ ways.

3C. PERMUTATIONS WITH REPEATED ITEMS

- If we arrange n objects, where n_1 are identical, n_2 are identical, ... n_k are identical, the total number of distinct arrangements is: $\frac{n!}{n_1!\times n_2!\times\ldots\times n_k!}$
- **Worked Example 5:** In how many ways can the letters of the word "MISSISSIPPI" be arranged?
  a. n **(total letters):** 11
  b. **Repeats:**
     - M: 1 ($n_1 = 1$)
     - I: 4 ($n_2 = 4$)
     - S: 4 ($n_3 = 4$)
     - P: 2 ($n_4 = 2$)
  c. **Formula:** Total = $\frac{11!}{1!\times4!\times4!\times2!}$
  d. **Solve:** $\frac{11\times10\times9\times8\times7\times6\times5\times(4!)}{(4!)\times(4\times3\times2\times1)\times(2\times1)}$
     - $= \frac{11\times10\times9\times8\times7\times6\times5}{24\times2} = 34{,}650$ ways.

3D. CIRCULAR PERMUTATIONS

- The number of ways to arrange n distinct objects in a **circle** is $(n-1)!$**.**
- **Reason:** In a circle, there is no "start" or "end." We "fix" one person's position, which breaks the circle into a line, and then arrange the remaining $(n-1)$ people.
- **Worked Example 6:** In how many ways can 7 people be seated at a round table?
  - $n = 7$.

- Ways = $(7-1)! = 6! = 720$ ways.

EVALUATION

1. What is the Fundamental Counting Principle?
2. What is the main difference between a permutation and a combination (which we will study next week)?
3. Evaluate $\frac{8!}{5!}$.
4. In how many ways can 6 students be arranged in a single line?
5. In how many distinct ways can the letters of the word "FURTHER" be arranged?

ASSIGNMENT

1. In how many ways can 5 boys and 4 girls be seated in a row such that:
   - (a) there are no restrictions?
   - (b) all the boys sit together and all the girls sit together?
   - (c) the boys and girls alternate?
2. How many 4-digit numbers can be formed from the digits 0, 1, 2, 3, 4, 5 if:
   - (a) repetition is not allowed?
   - (b) repetition is not allowed AND the number must be odd? (Hint: Start with the last digit).
3. In how many ways can 8 people be seated at a round table?
4. In how many ways can the letters of the word "MATHEMATICS" be arranged?
5. A shelf has 4 Physics books, 3 Chemistry books, and 2 Biology books. In how many ways can they be arranged if all books of the same subject must stay together?

WEEK 3: PERMUTATION AND COMBINATION II

1. COMBINATIONS

- **Definition:** A **combination** is a **selection** or **grouping** of objects from a set, where the **order of selection does not matter**.
- **Key Concept (Permutation vs. Combination):**
    - **Permutation:** Order matters. "Picking a President, VP, and Secretary." (A, B, C) is different from (B, A, C).
    - **Combination:** Order does not matter. "Picking a committee of 3." (A, B, C) is the *same* as (B, A, C).
- **Derivation:**
    - The number of *permutations* of r from n is ${}^{n}P_{r}$.
    - A set of r objects can be arranged in $r!$ ways.
    - Therefore, the number of *combinations* (selection) must be the number of permutations *divided* by the number of ways to arrange the selection.
    - $C(n,r) = \frac{{}^{n}P_{r}}{r!} = \frac{n!/(n-r)!}{r!} = \frac{n!}{r!(n-r)!}$
- **Formula:** $C(n,r)$ or ${}^{n}C_{r}$ or (most commonly in Further Maths) $\binom{n}{r}$ $\binom{n}{r} = \frac{n!}{r!(n-r)!}$
- **Symmetry Property:** $\binom{n}{r} = \binom{n}{n-r}$.
    - *Proof:* $\binom{n}{n-r} = \frac{n!}{(n-r)!(n-(n-r))!} = \frac{n!}{(n-r)!r!}$, which is the same.
    - *Logic:* The number of ways to *choose* 3 people for a team (from 10) is the same as the number of ways to *reject* 7 people.

2. WORKED EXAMPLES (COMBINATIONS)

- **Worked Example 1:** How many ways can a committee of 4 people be chosen from a group of 10 people?

    a. **Analyze:** Does order matter? No, a committee of (A, B, C, D) is the same as (D, C, B, A). It is a **combination**.

    b. $n = 10, r = 4$.

    c. **Solve:** $\binom{10}{4} = \frac{10!}{4!(10-4)!} = \frac{10!}{4!6!}$

        - $= \frac{10\times9\times8\times7\times(6!)}{(4\times3\times2\times1)\times(6!)}$
        - $= \frac{10\times9\times8\times7}{24}$
        - $(8 \times 3 = 24$, so they cancel 9×8 in the numerator)
        - $= 10 \times 3 \times 7 = 210$ ways.

- **Worked Example 2 (With Restrictions):** A committee of 5 is to be formed from 6 men and 4 women. In how many ways can this be done if the committee must contain (a) exactly 3 men, and (b) at least 3 men?

    - (a) **Exactly 3 men:**
        i. **Analyze:** We must choose 3 men from 6 *AND* 2 women from 4.
        ii. **Task 1 (Men):** $\binom{6}{3} = \frac{6!}{3!3!} = \frac{6\times5\times4}{3\times2\times1} = 20$ ways.
        iii. **Task 2 (Women):** $\binom{4}{2} = \frac{4!}{2!2!} = \frac{4\times3}{2\times1} = 6$ ways.
        iv. **Total (Counting Principle):** $20 \times 6 = 120$ ways.
    - (b) **At least 3 men:** This means we have *two* mutually exclusive cases.
        - **Case 1:** (3 Men AND 2 Women)
            - $\binom{6}{3} \times \binom{4}{2} = 20 \times 6 = 120$ ways.
        - **Case 2:** (4 Men AND 1 Woman)
            - $\binom{6}{4} \times \binom{4}{1} = \left(\frac{6!}{4!2!}\right) \times 4 = 15 \times 4 = 60$ ways.
        - **Case 3:** (5 Men AND 0 Women)

- $\binom{6}{5} \times \binom{4}{0} = 6 \times 1 = 6$ ways. (Note: $\binom{n}{0} = 1$).

- **Total (Addition Law):** We *add* the cases.
  - Total = $120 + 60 + 6 = 186$ ways.

---

3. APPLICATIONS IN PROBABILITY

Combination is the "counting" part of $n(E)$ and $n(S)$ when order doesn't matter.

- **Worked Example 3:** A bag contains 7 Red balls and 5 Blue balls. If 3 balls are drawn at random *without replacement*, what is the probability that *exactly 2* are Red?
  a. **Find $n(S)$ (Total Sample Space):**
     - We are choosing *any* 3 balls from a total of 12. Order doesn't matter.
     - $n(S) = \binom{12}{3} = \frac{12!}{3!9!} = \frac{12 \times 11 \times 10}{3 \times 2 \times 1} = 2 \times 11 \times 10 = 220$.
  b. **Find $n(E)$ (Favorable Outcomes):**
     - We need "exactly 2 Red" (which implies *AND* 1 Blue).
     - Choose 2 Red from 7: $\binom{7}{2} = \frac{7!}{2!5!} = \frac{7 \times 6}{2} = 21$ ways.
     - Choose 1 Blue from 5: $\binom{5}{1} = \frac{5!}{1!4!} = 5$ ways.
     - $n(E) = \binom{7}{2} \times \binom{5}{1} = 21 \times 5 = 105$.
  c. **Find Probability:**
     - $P(E) = \frac{n(E)}{n(S)} = \frac{105}{220} = \frac{21}{44}$.

EVALUATION

1. What is the main difference between a permutation and a combination?
2. Evaluate (a) $\binom{8}{3}$ and (b) $\binom{10}{7}$.
3. A student has to answer 8 out of 10 questions on an exam. How many ways can they choose the questions?

4. A committee of 3 men and 2 women is to be chosen from 7 men and 5 women. In how many ways can this be done?
5. What is the relationship $\binom{n}{r} = \binom{n}{n-r}$ called? Prove it using $\binom{9}{6}$.

ASSIGNMENT

1. From a group of 12 students, how many ways can a team of 8 be selected?
2. A bag contains 5 green and 7 yellow balls. 4 balls are drawn at random. Find the probability of selecting:
    - (a) 4 green balls.
    - (b) 2 green and 2 yellow balls.
    - (c) At least 1 green ball. (Hint: $1 - P(\text{no green balls})$).
3. How many different 5-card hands can be dealt from a standard deck of 52 cards?
4. A committee of 6 is to be formed from 5 teachers and 8 students. In how many ways can the committee be formed if:
    - (a) There are no restrictions?
    - (b) The committee must contain exactly 2 teachers?
    - (c) The committee must contain at least 4 students?
5. Prove that $\binom{n}{r} + \binom{n}{r-1} = \binom{n+1}{r}$ (Pascal's Identity).

WEEK 4: DYNAMICS I

1. INTRODUCTION TO DYNAMICS

Dynamics is the branch of mechanics that studies the *causes* of motion. It connects **Forces (F)** with the resulting **Acceleration (a)**. The entire subject is built upon Newton's Three Laws of Motion.

Key Definitions:

- **Force (F):** A push or a pull. A vector quantity. Unit: **Newton (N)**.
- **Mass (m):** The amount of "stuff" in an object; a measure of its **inertia** (resistance to change in motion). A scalar quantity. Unit: **Kilogram (kg)**.
- **Weight (W):** The force of gravity acting on a mass. $W = mg$, where $g \approx 9.8$ m/s^2 (or 10 m/s^2 for simplicity).
- **Net Force (F_{net}):** The *vector sum* of all forces acting on an object.

---

2. NEWTON'S LAWS OF MOTION

NEWTON'S FIRST LAW (THE LAW OF INERTIA)

- **Statement:** An object at rest will stay at rest, and an object in motion will stay in motion (with the same velocity/constant speed in a straight line), *unless* acted upon by an external **net force**.
- **Core Concept:** $F_{net} = 0 \Leftrightarrow a = 0$. (If the net force is zero, the acceleration is zero).
- **Example:** A book on a table. Gravity pulls it down (W), the table pushes it up (R). $W = R$, so $F_{net} = 0$, $a = 0$, and the book stays at rest.

NEWTON'S SECOND LAW (THE FUNDAMENTAL LAW)

- **Statement:** The **acceleration (a)** of an object is *directly proportional* to the **net force (F_{net})** acting on it and *inversely proportional* to its **mass (m)**.
- **The Equation:** This is the most important equation in classical mechanics. $F_{net} = ma$ (Net Force = mass × acceleration)

- **Worked Example 1:** A net force of 30 N acts on a 5 kg block. What is the acceleration?
    - $F_{net} = ma$
    - $30 = 5 \times a$
    - $a = 30/5 = 6$ m/s^2.
- **Worked Example 2:** A 10 kg box is pulled to the right by a force of 50 N. A friction force of 20 N opposes the motion. What is the acceleration?
    a. **Find Net Force:** F_{net} = Forward Force − Backward Force
        - $F_{net} = 50$ N $- 20$ N $= 30$ N (to the right).
    b. **Apply $F_{net} = ma$:**
        - $30 = 10 \times a$
        - $a = 3$ m/s^2 (to the right).

NEWTON'S THIRD LAW (ACTION-REACTION)

- **Statement:** For every action, there is an **equal and opposite reaction**.
- **Core Concept:** If object A exerts a force on object B, then object B simultaneously exerts an *equal* force in the *opposite* direction on object A.
- **Examples:**
    - A gun firing (Action: Gun pushes bullet forward. Reaction: Bullet pushes gun backward, "recoil").
    - Walking (Action: Your foot pushes the ground backward. Reaction: The ground pushes your foot forward).

---

3. MOTION ALONG AN INCLINED PLANE

This is a classic application of Newton's 2nd Law. The key is to use a **Free-Body Diagram (FBD)**.

The Free-Body Diagram (FBD):

1. **Weight ($W = mg$):** Always acts **vertically downwards**.

2. **Resolve the Weight:** We must "tilt" our axes (x-axis parallel to the plane, y-axis perpendicular). We resolve W into two components:
    - **Perpendicular Component:** $W_{\perp} = mg\cos(\theta)$. This is the force "pushing into" the plane.
    - **Parallel Component:** $W_{||} = mg\sin(\theta)$. This is the force "pulling down" the plane.
3. **Normal Reaction (R):** The force from the plane, acting *perpendicular* to the surface, pushing "out." $R = mg\cos(\theta)$ (if no other perpendicular forces).
4. **Friction (F_f):** Acts *parallel* to the plane, *opposing* the direction of motion. $F_f = \mu R = \mu mg\cos(\theta)$ (where μ is the coefficient of friction).

Equation of Motion: Apply $F_{net} = ma$ *along the plane*. If the object is sliding *down*: F_{net} = (Force Down) − (Force Up) = ma $\mathbf{mg\sin(\theta) - F_f = ma}$

- **Worked Example 3:** A 5 kg box slides down a ramp inclined at 30°. The coefficient of friction μ is 0.2. Find the acceleration. (Use $g = 10$ m/s^2).
    a. **Draw FBD and find forces:**
        - $m = 5$ kg, $\theta = 30°$, $\mu = 0.2$
    b. **Force pulling Down ($W_{||}$):**
        - $mg\sin(\theta) = (5)(10)\sin(30°) = 50 \times (0.5) = 25$ N.
    c. **Force of Friction (F_f):**
        - First, find Normal Reaction $R = mg\cos(\theta) = (5)(10)\cos(30°) = 50 \times (\sqrt{3}/2) \approx 43.3$ N.
        - $F_f = \mu R = 0.2 \times 43.3 = 8.66$ N.
    d. **Apply $F_{net} = ma$:**
        - F_{net} = (Force Down) − (Force Up)
        - $F_{net} = (mg\sin\theta) - (F_f)$
        - $ma = 25 - 8.66$
        - $5a = 16.34$
    e. **Solve for a:**

- $a = 16.34/5 = 3.268$ m/s^2.

EVALUATION

1. State Newton's Three Laws of Motion.
2. What is the "Net Force"? If a 100 kg object is moving at a *constant velocity* of 20 m/s, what is the net force on it? (Hint: Use 1st Law).
3. A 10 N force and a 6 N force act on a 4 kg block in opposite directions. Find the acceleration of the block.
4. A 2 kg block rests on a smooth (frictionless) plane inclined at 30°. What is its acceleration down the plane?
5. What is the difference between "mass" and "weight"? What is the weight of a 5 kg mass? (Use $g = 9.8$ m/s^2).

ASSIGNMENT

1. A car of mass 1200 kg, moving at 20 m/s, is brought to rest by a constant braking force in 5 seconds. Find (a) the car's deceleration, and (b) the magnitude of the braking force. (Hint: Use $v = u + at$).
2. Draw a free-body diagram of a 10 kg block being *pushed up* a rough inclined plane (at 30°) by a force P.
3. For the block in Q2, if the coefficient of kinetic friction μ_k is 0.3, what force P is required to push the block *up the plane at a constant velocity*? (Use $g = 10$ m/s^2).
4. A 20 kg block is pulled along a rough horizontal floor ($\mu_k = 0.4$) by a rope. The rope makes an angle of 30° *above* the horizontal. If the tension in the rope is 100 N, find the acceleration of the block.
5. What is "inertia"? Give a real-world example of Newton's 1st Law.

WEEK 5: MIDTERM EXAMINATION

This week is dedicated to the Midterm Examination.

- **Topics Covered:** All topics from Weeks 1–4.
    - **Week 1:** Probability rules (Addition, Multiplication), Mutually Exclusive and Independent/Dependent events.
    - **Week 2:** Fundamental Counting Principle, Factorials, Permutations ($n!$, ^nP_r), permutations with repeats, and circular permutations.
    - **Week 3:** Combinations ($\binom{n}{r}$), distinguishing combinations from permutations, and probability applications using combinations.
    - **Week 4:** Dynamics (Newton's 3 Laws, $F_{net} = ma$), Free-Body Diagrams, and solving problems on horizontal and inclined planes (with friction).
- **Format:** The test will include definitions, short-answer questions, probability problems, P&C counting problems, and dynamics problems (requiring FBDs and $F = ma$).

PAWILLS CONSULTING

WEEK 6: DYNAMICS II

1. TOPIC: MOTION OF CONNECTED PARTICLES

This is a direct application of Newton's Second Law ($F_{net} = ma$) to a *system* of two or more objects connected by a string (often passing over a pulley).

2. KEY ASSUMPTIONS (PULLEYS AND STRINGS)

In SS2 Further Maths, we make two critical assumptions:

1. The **string** is **"light"** (its mass is zero, so $W = 0$) and **"inextensible"** (its length never changes).
    - *Consequence:* The **acceleration (a)** is the *same* for all objects connected by the string.
2. The **pulley** is **"light"** (massless) and **"smooth"** (frictionless).
    - *Consequence:* The **Tension (T)** is *uniform* (the same) on both sides of the string.

3. THE STRATEGY FOR SOLVING

1. **Draw:** Draw a large, clear diagram of the system.
2. **Separate:** Draw a separate **Free-Body Diagram (FBD)** for *each* mass (particle) in the system. Show all forces (Weight, Tension, Reaction, Friction).
3. **Resolve:** For each FBD, apply $F_{net} = ma$ in the direction of motion.
4. **System:** This will create a *system of simultaneous equations* (usually two equations with two unknowns, a and T).
5. **Solve:** Solve the equations for a and T.

---

4. WORKED EXAMPLES

- **Worked Example 1: Two Masses on a Horizontal Table**

- A 3 kg block and a 2 kg block are on a smooth horizontal table, connected by a light string. A force of 20 N pulls the 3 kg block. Find the acceleration of the system and the tension in the string.

a. **System (Shortcut):** Treat both blocks as *one* 5 kg object.
   - $F_{net} = ma \Rightarrow 20 = (3+2)a \Rightarrow 20 = 5a \Rightarrow \mathbf{a = 4}$ m/s^2.

b. **Separate (To find Tension):**
   - **FBD for 3kg block:** The 20 N force pulls it *forward* (R), and the Tension (T) pulls it *backward* (L).
     - $F_{net} = ma \Rightarrow 20 - T = (3)(4) \Rightarrow 20 - T = 12 \Rightarrow T = 8$ N.
   - **FBD for 2kg block (Check):** The *only* force pulling it forward is the Tension (T).
     - $F_{net} = ma \Rightarrow T = (2)(4) \Rightarrow T = 8$ N.

- **Answer:** $a = 4$ m/s^2, $T = 8$ N.

- **Worked Example 2: Mass on Table, Mass Hanging (Attwood Machine)**
  - A 6 kg block (m_1) on a smooth horizontal table is connected by a string over a smooth pulley to a 4 kg block (m_2) hanging freely. Find the acceleration and tension. (Use $g = 10$ m/s^2).

a. **Analyze Motion:** m_2 (4kg) will pull m_1 (6kg) across the table. The whole system will accelerate.

b. **FBD for m_2 (4kg, hanging):**
   - Forces: Weight ($W_2 = 4g = 40$ N) pulls *down*. Tension (T) pulls *up*.
   - Motion is *down*.
   - $F_{net} = ma \Rightarrow$ (Down Force) $-$ (Up Force) $= m_2 a$
   - $40 - T = 4a$ --- (Eq 1)

c. **FBD for m_1 (6kg, table):**
   - Forces: Tension (T) pulls *right*. (Reaction/Weight cancel vertically).
   - Motion is *right*.
   - $F_{net} = ma \Rightarrow$ (Right Force) $-$ (Left Force) $= m_1 a$

- $T - 0 = 6a$
- $T = 6a$ --- (Eq 2)

d. **Solve Simultaneously:** Substitute (Eq 2) into (Eq 1).

- $40 - (6a) = 4a$
- $40 = 10a$
- $\mathbf{a = 4}$ m/s$^{\mathbf{2}}$

e. **Find T:** $T = 6a = 6(4) = \mathbf{24}$ N.

- **Worked Example 3: Two Hanging Masses (Attwood Machine)**
  - Two masses, $m_1 = 7$ kg and $m_2 = 3$ kg, are connected by a string over a smooth pulley. Find the acceleration and tension. (Use $g = 10$ m/s^2).

a. **Analyze Motion:** m_1 (7kg) is heavier, so it will accelerate *down*, and m_2 (3kg) will accelerate *up*.

b. **FBD for m_1 (7kg):**

- Forces: Weight ($W_1 = 7g = 70$ N) pulls *down*. Tension (T) pulls *up*.
- Motion is *down*.
- $F_{net} = ma \Rightarrow$ (Down Force) $-$ (Up Force) $= m_1 a$
- $70 - T = 7a$ --- (Eq 1)

c. **FBD for m_2 (3kg):**

- Forces: Tension (T) pulls *up*. Weight ($W_2 = 3g = 30$ N) pulls *down*.
- Motion is *up*.
- $F_{net} = ma \Rightarrow$ (Up Force) $-$ (Down Force) $= m_2 a$
- $T - 30 = 3a$ --- (Eq 2)

d. **Solve Simultaneously:** (This time, add the equations to eliminate T).

- (Eq 1): $70 - T = 7a$
- (Eq 2): $-30 + T = 3a$
- Add (1) + (2): $40 + 0 = 10a$
- $a = 40/10 = \mathbf{4}$ m/s$^{\mathbf{2}}$

e. **Find T:** (Use Eq 2) $T - 30 = 3(4) \Rightarrow T - 30 = 12 \Rightarrow T = \mathbf{42}$ N.

EVALUATION

1. What are the two key assumptions we make about a "light, inextensible string"?
2. What is the key assumption we make about a "smooth pulley"?
3. What is the 5-step strategy for solving any connected particle problem?
4. A 5 kg block and a 10 kg block are on a smooth table, connected by a string. A 30 N force pulls the 10 kg block. Find the acceleration a and the tension T in the string.
5. Two masses, 10 kg and 6 kg, hang over a smooth pulley. Find the acceleration of the system. (Use $g = 10$ m/s^2).

ASSIGNMENT

1. A 4 kg block on a *rough* horizontal table ($\mu_k = 0.5$) is connected by a string over a smooth pulley to a 6 kg block hanging freely. Find the acceleration and tension. (Use $g = 10$ m/s^2).
2. A 10 kg block (m_1) on a *smooth* inclined plane at 30° is connected by a string over a pulley to a 4 kg block (m_2) hanging freely.
    - (a) Draw the FBD for both masses.
    - (b) Find the acceleration of the system. (Hint: The 10kg block will pull the 4kg block *up*).
3. A 5 kg block and a 3 kg block are connected by a string. The 5 kg block is pulled by a 40 N force. The coefficient of friction for *both* blocks is 0.2. Find the acceleration a and the tension T.
4. In an Attwood machine, two masses of $m_1 = M$ and $m_2 = 2M$ are connected over a pulley. Find the acceleration a in terms of M and g.
5. Define: Tension, Inextensible, Smooth Pulley, Attwood Machine, System.

WEEK 7: MIDTERM BREAK

This week is for the Midterm Break. There are no formal lesson notes.

PAWILLS CONSULTING

WEEK 8: WORK, POWER, ENERGY, IMPULSE, AND MOMENTUM

1. WORK

- **Definition (Physics): Work** is done when a **force** F causes a **displacement** d. Work is *only* done by the component of the force that is *parallel* to the displacement.
- **Formula:** $W = F \times d$
- **Unit:** The **Joule (J)**. (1 Joule = 1 Newton-meter)
- **Work Done by an Angled Force:** If a force F pulls at an angle θ to the horizontal, the component doing the work is $F\cos(\theta)$.
  - $W = (F\cos\theta) \times d = Fd\cos\theta$
  - (This is the **dot product**: $W = \mathbf{F} \cdot \mathbf{d}$)
- **Work Done Against Gravity:** When you lift an object of mass m to a vertical height h, the force you apply is equal to its weight (mg).
  - $W = F \times d = (mg) \times h = mgh.$
  - This work is *stored* as Potential Energy.

---

2. ENERGY

- **Definition: Energy** is the **capacity to do work**.
- **Unit:** The **Joule (J)**.
- **Forms:**
  - **Kinetic Energy (K.E.):** The energy of *motion*. $K.E. = \frac{1}{2}mv^2$
  - **Potential Energy (P.E.):** The *stored* energy of position. $P.E. = mgh$

2A. WORK-ENERGY THEOREM

This theorem provides a direct link between Work and Energy.

- **Statement:** The **net work done** on an object is equal to the **change in its kinetic energy**. $W_{net} = \Delta K.E. = K.E._{\text{final}} - K.E._{\text{initial}}$ $W_{net} = \frac{1}{2}mv^2 - \frac{1}{2}mu^2$

- **Worked Example 1:** A 4 kg block, starting from rest, is pulled by a net force of 10 N over a distance of 5 m. What is its final speed?
    - **Method A (F=ma):** $F = ma \Rightarrow 10 = 4a \Rightarrow a = 2.5$ m/s^2.
        - $v^2 = u^2 + 2as \Rightarrow v^2 = 0^2 + 2(2.5)(5) = 25 \Rightarrow v = 5$ m/s.
    - **Method B (Work-Energy):**
        - $W_{net} = F \times d = 10$ N $\times$ 5 m $= 50$ J.
        - $W_{net} = \Delta K.E. = \frac{1}{2}mv^2 - \frac{1}{2}mu^2$
        - $50 = \frac{1}{2}(4)v^2 - 0$
        - $50 = 2v^2 \Rightarrow v^2 = 25 \Rightarrow v = 5$ m/s.

2B. CONSERVATION OF ENERGY

- **Statement:** In an isolated system (with no friction/air resistance), the **total mechanical energy** ($E = K.E. + P.E.$) is conserved (remains constant). $P.E._{\text{initial}} + K.E._{\text{initial}} = P.E._{\text{final}} + K.E._{\text{final}}$ $mgh_i + \frac{1}{2}mu_i^2 = mgh_f + \frac{1}{2}mu_f^2$
- **Worked Example 2:** A 2 kg ball is dropped from a height of 10 m. What is its speed just before it hits the ground? (Use $g = 10$ m/s^2).
    a. **Initial State (top):** $h_i = 10$m, $u_i = 0$ (dropped).
    b. **Final State (bottom):** $h_f = 0$m, $v_f = ?$
    c. **Apply Conservation:**
        - $(mgh_i + \frac{1}{2}mu_i^2) = (mgh_f + \frac{1}{2}mu_f^2)$
        - $(2)(10)(10) + 0 = 0 + \frac{1}{2}(2)v_f^2$
        - $200 = v_f^2$
        - $v_f = \sqrt{200} \approx 14.14$ m/s.

---

3. POWER

- **Definition: Power** is the **rate** at which work is done (or energy is transferred).

- **Unit:** The **Watt (W)**. (1 Watt = 1 Joule per second).
- **Formulas:** $P = \frac{W}{t}$ (Power = Work / time) Since $W = F \times d$: $P = \frac{F \times d}{t} = F \times \left(\frac{d}{t}\right)$ Since $v = d/t$: $P = F \times v$ (Power = Force × velocity)
- **Worked Example 3:** A 60 kg student runs up a flight of stairs with a vertical height of 5 m in 4 seconds. What is their average power output? (Use $g = 10$ m/s^2).
    a. **Work Done:** The work is done against gravity.
        - $W = mgh = (60)(10)(5) = 3000$ J.
    b. **Power:**
        - $P = W/t = 3000$ J$/4$ s $= 750$ W.

---

4. MOMENTUM AND IMPULSE

- **Momentum (p):** A measure of an object's "quantity of motion." It is a vector.
    - **Formula:** $p = mv$ (mass × velocity)
    - **Unit:** kg m/s
- **Impulse (I):** The *change* in momentum (Δp). Impulse is also equal to the force applied multiplied by the time it was applied.
    - **Formula:** $I = F \times t = \Delta p = mv - mu$
    - **Unit:** Ns (or kg m/s)
- **Conservation of Momentum:** In a closed system (with no external forces), the **total momentum *before* a collision is equal to the total momentum *after* the collision.** $m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$
- **Worked Example 4:** A 10 kg truck moving at 5 m/s (u1) collides with a 5 kg car at rest (u2=0). They stick together. What is their common final velocity (v)?
    a. **Apply Conservation:**
        - $m_1u_1 + m_2u_2 = (m_1 + m_2)v$ (since they stick)

b. **Substitute:**

- $(10)(5) + (5)(0) = (10 + 5)v$
- $50 + 0 = 15v$
- $v = 50/15 = 10/3$ m/s.

o **Answer:** $v \approx 3.33$ m/s.

EVALUATION

1. Define Work, Power, and Energy. State the SI unit for each.
2. A 1000 kg car accelerates from 5 m/s to 15 m/s. What is the *net work done* on the car? (Hint: Use Work-Energy Theorem).
3. A 50 N force pulls a box at a constant speed of 4 m/s. What is the *power* being supplied?
4. A 2 kg ball traveling at 10 m/s hits a wall and rebounds at 8 m/s. What is the *impulse* (change in momentum)? (Hint: Velocity is a vector).
5. A 3 kg ball is dropped from a height h. It hits the ground with a speed of 20 m/s. Find the initial height h (Use $g = 10$ m/s^2).

ASSIGNMENT

1. An engine is rated at 50 kW. How much work can it do in 20 seconds?
2. A 100 g bullet (0.1 kg) traveling at 500 m/s strikes a wooden block and stops. What is the change in the bullet's K.E.?
3. A 70 kg man runs up a 10 m high staircase. If his power output is 1400 W, how long does it take him? (Use $g = 10$ m/s^2).
4. A 2 kg cart (A) moving at 5 m/s collides with a 3 kg cart (B) moving at 2 m/s *in the same direction*. After the collision, cart A moves at 3 m/s. What is the new velocity of cart B?
5. Define: Work-Energy Theorem, Conservation of Energy, Conservation of Momentum, Impulse.

WEEK 9: PROJECTILES

1. INTRODUCTION

A **projectile** is any object that is thrown, launched, or "projected" into the air and is then subject *only* to the force of **gravity** (and negligible air resistance).

The Key Principle: We must analyze the motion by separating it into two **independent** components:

1. **Horizontal Motion (x-direction):** There are *no forces* in this direction (since we ignore air resistance).
    - Acceleration (a_x) = **0**
    - Velocity (v_x) = **Constant**
2. **Vertical Motion (y-direction):** The *only* force is gravity.
    - Acceleration (a_y) = $-g$ (where $g \approx 9.8$ m/s^2)
    - Velocity changes due to gravity.

---

2. DERIVATION OF PROJECTILE EQUATIONS

Let's launch a projectile from the origin $(0,0)$ with an initial velocity u at an angle θ to the horizontal.

- **Initial Velocity Components:**
    - Initial horizontal velocity: $\mathbf{u_x = u\cos(\theta)}$
    - Initial vertical velocity: $\mathbf{u_y = u\sin(\theta)}$

2A. EQUATIONS AT TIME t

- **Horizontal Position (x):**
    - $x = u_x t = \mathbf{(u\cos\theta)t}$
- **Vertical Position (y):** (using $s = ut + \frac{1}{2}at^2$)

- $y = u_y t + \frac{1}{2} a_y t^2 = (\mathbf{usin\theta})\mathbf{t} - \frac{\mathbf{1}}{\mathbf{2}}\mathbf{gt^2}$

- **Horizontal Velocity (v_x):**
  - $v_x = u_x = \mathbf{ucos\theta}$ (It is constant!)
- **Vertical Velocity (v_y):** (using $v = u + at$)
  - $v_y = u_y + a_y t = \mathbf{usin\theta - gt}$

2B. KEY PARAMETERS

3. **Time to Maximum Height (t_h):**
   - At the *maximum height*, the projectile stops moving up, so its *vertical velocity* (v_y) is **0**.
   - $v_y = u\sin\theta - gt_h = 0$
   - $u\sin\theta = gt_h$
   - $t_h = \frac{u\sin\theta}{g}$
4. **Time of Flight (T):**
   - This is the *total* time the projectile is in the air. (Assuming it lands at the same height it started).
   - The time to go up (t_h) is equal to the time to come down.
   - $T = 2 \times t_h$
   - $T = \frac{2u\sin\theta}{g}$
5. **Maximum Height (H):**
   - This is the y position when $t = t_h$.
   - $H = (u\sin\theta)t_h - \frac{1}{2} g t_h^2$
   - $H = (u\sin\theta)\left(\frac{u\sin\theta}{g}\right) - \frac{1}{2} g \left(\frac{u\sin\theta}{g}\right)^2$
   - $H = \frac{u^2\sin^2\theta}{g} - \frac{1}{2} g \left(\frac{u^2\sin^2\theta}{g^2}\right)$
   - $H = \frac{u^2\sin^2\theta}{g} - \frac{u^2\sin^2\theta}{2g}$
   - $H = \frac{u^2\sin^2\theta}{2g}$

LAWILLS CONSULTING

6. **Range (R):**
    - This is the *total horizontal distance* traveled in the *total time* (T).
    - $R = x$ when $t = T$.
    - $R = (u\cos\theta) \times T = (u\cos\theta) \times \left(\frac{2u\sin\theta}{g}\right)$
    - $R = \frac{u^2(2\sin\theta\cos\theta)}{g}$
    - Using the trig identity: $\sin(2\theta) = 2\sin\theta\cos\theta$
    - $R = \frac{u^2\sin(2\theta)}{g}$
    - **Max Range:** The range R is maximum when $\sin(2\theta) = 1$, which happens when $2\theta = 90°$, or $\boldsymbol{\theta = 45°}$.

---

3. WORKED EXAMPLES

- **Worked Example 1:** A ball is kicked from the ground with an initial velocity of 20 m/s at an angle of 30° to the horizontal. (Use $g = 10$ m/s^2). Find:
    - (a) The time to reach maximum height.
    - (b) The maximum height reached.
    - (c) The total time of flight.
    - (d) The horizontal range.

Find Initial Components:

- $u = 20$ m/s, $\theta = 30°$
- $u_x = 20\cos(30°) = 20(\sqrt{3}/2) = 10\sqrt{3}$ m/s.
- $u_y = 20\sin(30°) = 20(1/2) = 10$ m/s.

(a) Time to Max Height (t_h):

- $t_h = \frac{u\sin\theta}{g} = \frac{10}{10} = \mathbf{1}$ s.

(b) Max Height (H):

- $H = \frac{u^2\sin^2\theta}{2g} = \frac{(10)^2}{2(10)} = \frac{100}{20} = \mathbf{5}$ m.

(c) Time of Flight (T):

- $T = 2 \times t_h = 2 \times 1 = \mathbf{2}$ s.

(d) Range (R):

- $R = u_x \times T = (10\sqrt{3}) \times 2 = 20\sqrt{3}$ m
- *or* $R = \frac{u^2 \sin(2\theta)}{g} = \frac{20^2 \sin(60°)}{10} = \frac{400(\sqrt{3}/2)}{10} = 20\sqrt{3}$ m.
- $R \approx \mathbf{34.64}$ m.

EVALUATION

1. What is the *key principle* for analyzing projectile motion?
2. What is the acceleration of a projectile in the (a) horizontal (x) direction and (b) vertical (y) direction?
3. At what point in its flight is a projectile's *vertical* velocity zero?
4. At what launch angle (θ) is the horizontal range (R) a maximum?
5. A stone is thrown with a velocity of 30 m/s at an angle of 45°. Find its total time of flight. (Use $g = 10$ m/s^2).

ASSIGNMENT

1. A cannonball is fired with an initial velocity of 100 m/s at an angle of 60° to the horizontal. (Use $g = 10$ m/s^2).
    - (a) Find the time it takes to reach its highest point.
    - (b) Find the maximum height it reaches.
    - (c) Find its horizontal range.
2. Prove that the maximum range of a projectile is given by $R_{\text{max}} = u^2/g$.
3. A ball is thrown *horizontally* from the top of a 80m high cliff with a speed of 10 m/s.
    - (a) How long does it take to hit the ground? (Hint: $y = u_y t - \frac{1}{2} g t^2$. $u_y = 0$).
    - (b) How far from the base of the cliff does it land?
4. Derive the "Equation of the Trajectory" (an equation for y in terms of x) for a projectile.
5. What is "air resistance," and how does it (in the real world) affect the path of a projectile compared to the "ideal" parabolic path?

PAWILLS CONSULTING

WEEK 10: BINOMIAL EXPANSION

1. INTRODUCTION

The **Binomial Theorem** provides a fast, systematic method for expanding (multiplying out) a binomial (a two-term expression) raised to a power, $(a + b)^n$, without doing the tedious, repetitive multiplication.

---

2. PASCAL'S TRIANGLE

For small, positive integer values of n, we can use Pascal's Triangle to find the **coefficients** of the expansion.

n=0: 1
n=1: 1 1
n=2: 1 2 1
n=3: 1 3 3 1
n=4: 1 4 6 4 1
n=5: 1 5 10 10 5 1
...

- **Worked Example 1:** Expand $(a + b)^4$.
    a. **Get coefficients (n=4):** 1, 4, 6, 4, 1.
    b. **Write powers of a:** Start at a^4 and descend: a^4, a^3, a^2, a^1, a^0.
    c. **Write powers of b:** Start at b^0 and ascend: b^0, b^1, b^2, b^3, b^4.
    d. **Combine:** $(a + b)^4 = 1a^4b^0 + 4a^3b^1 + 6a^2b^2 + 4a^1b^3 + 1a^0b^4$ $(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$
- **Worked Example 2:** Expand $(x - 2y)^3$.
    e. **Identify:** $a = x$, $b = -2y$, $n = 3$.
    f. **Coefficients (n=3):** 1, 3, 3, 1.

g. **Combine:**

- $1(x)^3(-2y)^0 = 1 \cdot x^3 \cdot 1 = x^3$
- $3(x)^2(-2y)^1 = 3 \cdot x^2 \cdot (-2y) = -6x^2y$
- $3(x)^1(-2y)^2 = 3 \cdot x \cdot (4y^2) = 12xy^2$
- $1(x)^0(-2y)^3 = 1 \cdot 1 \cdot (-8y^3) = -8y^3$

o **Answer:** $(x-2y)^3 = \mathbf{x^3 - 6x^2y + 12xy^2 - 8y^3}$

---

3. THE BINOMIAL THEOREM (GENERAL FORM)

Pascal's Triangle is slow for large n. The general theorem uses **Combinations ($\binom{n}{r}$)** for the coefficients.

$(a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^{n-r}b^r$ $(a+b)^n = \binom{n}{0}a^nb^0 + \binom{n}{1}a^{n-1}b^1 + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{n}a^0b^n$

THE GENERAL TERM

This is the most powerful part. The $(r+1)^{th}$ term of the expansion $(a+b)^n$ is: $T_{r+1} = \binom{n}{r}a^{n-r}b^r$

- **Worked Example 3:** Find the 5th term in the expansion of $(x+2y)^8$.
  a. **Analyze:** We want the 5th term, so $r+1=5 \Rightarrow r=4$.
  b. **Identify:** $n=8$, $a=x$, $b=2y$, $r=4$.
  c. **Formula:** $T_{r+1} = \binom{n}{r}a^{n-r}b^r$
  d. $T_5 = \binom{8}{4}(x)^{8-4}(2y)^4$
  e. **Calculate $\binom{8}{4}$:** $\frac{8!}{4!4!} = \frac{8\times7\times6\times5}{4\times3\times2\times1} = 70$.
  f. **Simplify:** $T_5 = 70 \cdot x^4 \cdot (16y^4) = 1120x^4y^4$.
  o **Answer:** The 5th term is $1120x^4y^4$.
- **Worked Example 4:** Find the term independent of x (the constant term) in the expansion of $\left(x^2 + \frac{1}{x}\right)^9$.

a. **Analyze:** We need to find the value of r that makes the powers of x cancel out to x^0.

b. **Identify:** $n = 9$, $a = x^2$, $b = x^{-1}$.

c. **Set up the powers from T_{r+1}:**

- $(a^{n-r})(b^r) \Rightarrow (x^2)^{9-r}(x^{-1})^r$

d. **Solve for r:** We want the final power to be 0.

- $x^{2(9-r)} \cdot x^{-r} = x^0$
- $x^{18-2r-r} = x^0$
- $18 - 3r = 0 \Rightarrow 3r = 18 \Rightarrow r = 6$.

e. **Find the term:** The term we need is the $T_{r+1} = T_7$ term.

- $T_7 = \binom{9}{6}(x^2)^{9-6}\left(\frac{1}{x}\right)^6$
- $\binom{9}{6} = \binom{9}{3} = \frac{9\times8\times7}{3\times2\times1} = 84$.
- $T_7 = 84 \cdot (x^2)^3 \cdot (x^{-6}) = 84 \cdot x^6 \cdot x^{-6} = 84x^0 = 84$.

- **Answer:** The term independent of x is **84**.

EVALUATION

1. Write out the $n = 6$ row of Pascal's Triangle.
2. Expand $(x + y)^5$ using Pascal's Triangle.
3. Expand $(2a - b)^4$ using the Binomial Theorem.
4. What is the formula for the general term T_{r+1} of $(a + b)^n$?
5. Find the 3rd term in the expansion of $(x + 5y)^7$.

ASSIGNMENT

1. Find the 6th term in the expansion of $(2x - y^2)^9$.
2. Find the coefficient of the term in x^4 in the expansion of $(1 + 2x)^8$.
3. Use the Binomial Theorem to find the value of $(1.02)^5$ correct to 4 decimal places. (Hint: $(1 + 0.02)^5$).

PAWELS CONSULTING

4. Find the term independent of x in the expansion of $\left(x-\frac{2}{x^2}\right)^{12}$.

5. Find the middle term(s) in the expansion of $\left(3x+\frac{y}{2}\right)^{8}$.

WEEK 11: MECHANICS (VECTOR GEOMETRY)

1. INTRODUCTION TO VECTORS

- **Scalar:** A quantity with **magnitude (size) only**. (e.g., mass (5kg), speed (10 m/s), time (3s)).
- **Vector:** A quantity with both **magnitude and direction**. (e.g., velocity (10 m/s *North*), force (50N *downwards*)).
- **Vector Notation:**
  - **3D Cartesian:** $\mathbf{a} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$
    - $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are the **unit vectors** (length 1) in the x, y, z directions.
  - **Column Vector:** $\mathbf{a} = \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix}$

---

2. VECTOR OPERATIONS

Let $\mathbf{a} = 3\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$ and $\mathbf{b} = \mathbf{i} - 5\mathbf{j} + 3\mathbf{k}$.

- **Addition/Subtraction:** Add/subtract corresponding components.
  - $\mathbf{a} + \mathbf{b} = (3 + 1)\mathbf{i} + (4 - 5)\mathbf{j} + (-2 + 3)\mathbf{k} = \mathbf{4i - j + k}$
  - $\mathbf{a} - \mathbf{b} = (3 - 1)\mathbf{i} + (4 - (-5))\mathbf{j} + (-2 - 3)\mathbf{k} = \mathbf{2i + 9j - 5k}$
- **Scalar Multiplication:** Multiply each component by the scalar.
  - $3\mathbf{a} = 3(3\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}) = \mathbf{9i + 12j - 6k}$
- **Magnitude (Modulus):** The *length* of the vector. (Pythagoras in 3D).
  - $|\mathbf{a}| = \sqrt{a_x^2 + a_y^2 + a_z^2}$
  - $|\mathbf{a}| = \sqrt{3^2 + 4^2 + (-2)^2} = \sqrt{9 + 16 + 4} = \sqrt{29}$
- **Unit Vector:** A vector of length 1 in the same direction. $\hat{\mathbf{a}} = \frac{\mathbf{a}}{|\mathbf{a}|}$.
  - $\hat{\mathbf{a}} = \frac{3\mathbf{i}+4\mathbf{j}-2\mathbf{k}}{\sqrt{29}}$

3. VECTOR PRODUCTS

3A. THE SCALAR (DOT) PRODUCT

- **Definition:** The product of two vectors that results in a **scalar** (a number).
- **Formulas:**
  a. Geometric: $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos(\theta)$ (where θ is the angle between them)
  b. Component: $\mathbf{a} \cdot \mathbf{b} = a_x b_x + a_y b_y + a_z b_z$
- **Key Properties:**
  - **Perpendicular Test:** If $\mathbf{a} \cdot \mathbf{b} = 0$, then $\mathbf{a}$ and $\mathbf{b}$ are **perpendicular** (orthogonal).
  - $\mathbf{i} \cdot \mathbf{i} = 1, \mathbf{j} \cdot \mathbf{j} = 1, \mathbf{k} \cdot \mathbf{k} = 1$
  - $\mathbf{i} \cdot \mathbf{j} = 0, \mathbf{j} \cdot \mathbf{k} = 0, \mathbf{i} \cdot \mathbf{k} = 0$
- **Application in Mechanics:**
  - **Work Done (W)** is the dot product of Force ($\mathbf{F}$) and Displacement ($\mathbf{d}$).
  - $W = \mathbf{F} \cdot \mathbf{d} = Fd\cos(\theta)$
- **Worked Example 1:** Let $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$ and $\mathbf{b} = 4\mathbf{i} - 2\mathbf{j} + 5\mathbf{k}$.
  - (a) Find $\mathbf{a} \cdot \mathbf{b}$.
    - $\mathbf{a} \cdot \mathbf{b} = (2)(4) + (3)(-2) + (-1)(5) = 8 - 6 - 5 = \mathbf{-3}$.
  - (b) A force $\mathbf{F} = (2\mathbf{i} + 3\mathbf{j} - \mathbf{k})$ N moves an object through a displacement $\mathbf{d} = (4\mathbf{i} - 2\mathbf{j} + 5\mathbf{k})$ m. Find the work done.
    - $W = \mathbf{F} \cdot \mathbf{d} = (2)(4) + (3)(-2) + (-1)(5) = 8 - 6 - 5 = \mathbf{-3}$ Joules.
  - (c) Find the angle θ between $\mathbf{a}$ and $\mathbf{b}$.
    - $\cos(\theta) = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$
    - $\mathbf{a} \cdot \mathbf{b} = -3$
    - $|\mathbf{a}| = \sqrt{2^2 + 3^2 + (-1)^2} = \sqrt{4 + 9 + 1} = \sqrt{14}$
    - $|\mathbf{b}| = \sqrt{4^2 + (-2)^2 + 5^2} = \sqrt{16 + 4 + 25} = \sqrt{45}$

TAWILLS CONSULTING

- $\cos(\theta) = \frac{-3}{\sqrt{14}\sqrt{45}} = \frac{-3}{\sqrt{630}} \approx -0.1195$
- $\theta = \arccos(-0.1195) \approx \mathbf{96.86}^\circ$

3B. THE VECTOR (CROSS) PRODUCT

- **Definition:** The product of two vectors that results in a new **vector** that is **perpendicular** to *both* original vectors.
- **Formula (Determinant Method):** $\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$
- **Application in Mechanics:**
  - **Moment (Torque, τ)** is the cross product of the position vector ($\mathbf{r}$) and the Force ($\mathbf{F}$).
  - $\tau = \mathbf{r} \times \mathbf{F}$
- **Worked Example 2:** Let $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$ and $\mathbf{b} = 4\mathbf{i} - 2\mathbf{j} + 5\mathbf{k}$. Find $\mathbf{a} \times \mathbf{b}$.
  - $\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & -1 \\ 4 & -2 & 5 \end{vmatrix}$
  - $= \mathbf{i}\begin{vmatrix} 3 & -1 \\ -2 & 5 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 2 & -1 \\ 4 & 5 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 2 & 3 \\ 4 & -2 \end{vmatrix}$
  - $= \mathbf{i}((3)(5) - (-1)(-2)) - \mathbf{j}((2)(5) - (-1)(4)) + \mathbf{k}((2)(-2) - (3)(4))$
  - $= \mathbf{i}(15 - 2) - \mathbf{j}(10 + 4) + \mathbf{k}(-4 - 12)$
  - $= \mathbf{13i - 14j - 16k}$

EVALUATION

1. What is the difference between a scalar and a vector?
2. Given $\mathbf{a} = 5\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ and $\mathbf{b} = 2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}$. Find:
   - (a) $2\mathbf{a} - \mathbf{b}$
   - (b) The magnitude $|\mathbf{a}|$.
3. For the vectors in Q2, find the scalar product $\mathbf{a} \cdot \mathbf{b}$.

PAUL'S CONSULTING

4. A force $\mathbf{F} = (4\mathbf{i} + \mathbf{j})$ N moves an object from the origin to the point $\mathbf{r} = (3\mathbf{i} + 5\mathbf{j})$ m. What is the work done?
5. Find the cross product $\mathbf{a} \times \mathbf{b}$ for the vectors in Q2.

ASSIGNMENT

1. Find a unit vector in the direction of $\mathbf{v} = 4\mathbf{i} - 3\mathbf{j} + 12\mathbf{k}$.
2. Given $\mathbf{a} = 2\mathbf{i} + p\mathbf{j} + \mathbf{k}$ and $\mathbf{b} = 4\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$. Find the value of p if $\mathbf{a}$ and $\mathbf{b}$ are perpendicular.
3. Find the angle between the vectors $\mathbf{a} = \mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{b} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$.
4. Find $\mathbf{a} \times \mathbf{b}$ if $\mathbf{a} = (1,0,3)$ and $\mathbf{b} = (2,5,-1)$.
5. The velocity of a particle is $\mathbf{v} = (3\mathbf{i} + 4\mathbf{j})$ m/s and its mass is 2 kg. Find its (a) speed and (b) momentum vector.

PAWILLS CONSULTING

WEEK 12: INTEGRATION I

1. DEFINITION (INDEFINITE INTEGRAL)

Integration is the **reverse process of differentiation**. It is also known as finding the **anti-derivative**.

- If $\frac{d}{dx}(x^3) = 3x^2$,
- Then the **integral** of $3x^2$ is x^3.
- **Problem:** $\frac{d}{dx}(x^3 + 5) = 3x^2$ and $\frac{d}{dx}(x^3 - 100) = 3x^2$.
- When we "reverse" $3x^2$, we don't know what the original constant was. We represent this unknown constant with "$+\mathbf{C}$".
- The **indefinite integral** finds the *family* of all possible anti-derivatives.

---

2. BASIC RULES OF INTEGRATION

2A. THE POWER RULE (REVERSE)

- **Rule:** $\int x^n dx = \frac{x^{n+1}}{n+1} + C \quad (\text{for } n \neq -1)$
- **Mnemonic:** "Add one to the power, then divide by the new power."
- **Worked Example 1:** Find $\int x^4 dx$.
    - $n = 4$. New power is $4 + 1 = 5$.
    - $\int x^4 dx = \frac{x^5}{5} + C$.
- **Worked Example 2:** Find $\int (3x^2 - 6x + 2) dx$.
    - We integrate *term by term*.
    - $\int 3x^2 dx = 3\left(\frac{x^3}{3}\right) = x^3$
    - $\int -6x dx = -6\left(\frac{x^2}{2}\right) = -3x^2$

- $\int 2dx = \int 2x^0 dx = 2\left(\frac{x^1}{1}\right) = 2x$
- **Answer:** $x^3 - 3x^2 + 2x + C$. (You only need one $+C$ for the whole expression).

- **Worked Example 3 (Fractional/Negative):** Find $\int (4\sqrt{x} + \frac{6}{x^3})dx$.

a. **Rewrite:** $\int (4x^{1/2} + 6x^{-3})dx$

b. **Integrate $4x^{1/2}$:** $n = 1/2$. New power = $1/2 + 1 = 3/2$.

- $4\left(\frac{x^{3/2}}{3/2}\right) = 4 \cdot \frac{2}{3}x^{3/2} = \frac{8}{3}x^{3/2}$

c. **Integrate $6x^{-3}$:** $n = -3$. New power = $-3 + 1 = -2$.

- $6\left(\frac{x^{-2}}{-2}\right) = -3x^{-2}$

d. **Combine:** $\frac{8}{3}x^{3/2} - 3x^{-2} + C$

- **Answer (simplified):** $\frac{8}{3}\sqrt{x^3} - \frac{3}{x^2} + C$.

2B. SPECIAL INTEGRALS (REVERSE OF WEEK 4)

- **The $n = -1$ Case:** $\int x^{-1}dx = \int \frac{1}{x}dx = \ln|x| + C$
- **Exponential:** $\int e^x dx = e^x + C \quad \Rightarrow \quad \int e^{kx}dx = \frac{1}{k}e^{kx} + C$
- **Trigonometric:**
  - $\int \cos(x)dx = \sin(x) + C$
  - $\int \sin(x)dx = -\cos(x) + C$
  - $\int \sec^2(x)dx = \tan(x) + C$
- **Worked Example 4:** Find $\int (5e^{3x} + \cos(2x))dx$.
  - $\int 5e^{3x}dx = 5\left(\frac{1}{3}e^{3x}\right) = \frac{5}{3}e^{3x}$
  - $\int \cos(2x)dx = \frac{1}{2}\sin(2x)$
  - **Answer:** $\frac{5}{3}e^{3x} + \frac{1}{2}\sin(2x) + C$

PAWILLS CONSULTING

3. APPLICATIONS (FINDING THE CONSTANT C)

The indefinite integral gives a *family* of functions. If we are given an **initial condition** (a point on the curve), we can find the *specific* function.

- **Worked Example 5:** The gradient of a curve is given by $\frac{dy}{dx} = 3x^2 - 4x$. The curve passes through the point $(2,5)$. Find the equation of the curve.
    a. **Integrate to find y:** y is the anti-derivative of dy/dx.
        - $y = \int (3x^2 - 4x)dx$
        - $y = 3\left(\frac{x^3}{3}\right) - 4\left(\frac{x^2}{2}\right) + C$
        - $y = x^3 - 2x^2 + C$
    b. **Use the point $(2,5)$ to find C:**
        - Substitute $x = 2$ and $y = 5$.
        - $5 = (2)^3 - 2(2)^2 + C$
        - $5 = 8 - 2(4) + C$
        - $5 = 8 - 8 + C \Rightarrow C = 5$.
    c. **Write the final equation:**
        - **Answer:** $y = x^3 - 2x^2 + 5$.

EVALUATION

1. What is an "indefinite integral," and why must you always add "+ C"?
2. Find $\int (8x^3 - 6x^2 + 2x + 1)dx$.
3. Find $\int (12\sqrt[3]{x} + \frac{5}{x^4})dx$.
4. Find $\int (2\sin x + 3\cos x)dx$.
5. The gradient of a curve is $\frac{dy}{dx} = 4x + 1$. If the curve passes through the origin $(0,0)$, find its equation.

ASSIGNMENT

1. Find the integral of $y = (x + 1)(2x - 3)$. (Hint: Expand first).

2. Find $\int (\frac{x^3+4x}{x^2})dx$. (Hint: Simplify first).
3. Find $\int (e^{4x} + \frac{1}{x})dx$.
4. The acceleration of a particle is $a(t) = 6t - 2$.
    - (a) Find the equation for its velocity $v(t)$, given that its initial velocity $v(0) = 5$.
    - (b) Find the equation for its displacement $s(t)$, given that its initial displacement $s(0) = 1$.
5. Find $\int (\sin(3x) + \cos(x/2))dx$.

PAWILLS CONSULTING

WEEK 13: INTEGRATION II AND APPLICATIONS

1. THE DEFINITE INTEGRAL

- **Definition:** The **definite integral** calculates the integral of a function between two limits (a "lower limit" a and an "upper limit" b).
- **The Fundamental Theorem of Calculus:** $\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a)$
    - Where $F(x)$ is the anti-derivative (indefinite integral) of $f(x)$.
    - **Note:** The "+ C" is not needed, as it would just cancel out: $(F(b) + C) - (F(a) + C) = F(b) - F(a)$.
- **Worked Example 1:** Evaluate $\int_1^3 (3x^2 + 2)dx$.
    a. **Find the anti-derivative:**
        - $F(x) = \int (3x^2 + 2)dx = x^3 + 2x$
    b. **Apply the limits $F(b) - F(a)$:**
        - $[x^3 + 2x]_1^3$
        - $= F(3) - F(1)$
        - $= ((3)^3 + 2(3)) - ((1)^3 + 2(1))$
        - $= (27 + 6) - (1 + 2)$
        - $= 33 - 3 = 30.$
    - **Answer:** 30.

2. APPLICATION 1: AREA UNDER A CURVE

The definite integral $\int_a^b f(x)dx$ gives the *exact* area bounded by the curve $y = f(x)$, the x-axis, and the vertical lines $x = a$ and $x = b$.

- **Worked Example 2:** Find the area under the curve $y = x^2 + 1$ from $x = 0$ to $x = 2$.
    a. **Set up the integral:**

- $A = \int_0^2 (x^2 + 1)dx$

b. **Find the anti-derivative:**

- $F(x) = \frac{x^3}{3} + x$

c. **Evaluate:**

- $A = [\frac{x^3}{3} + x]_0^2$
- $= F(2) - F(0)$
- $= \left(\frac{(2)^3}{3} + 2\right) - \left(\frac{(0)^3}{3} + 0\right)$
- $= \left(\frac{8}{3} + 2\right) - 0$
- $= \frac{8}{3} + \frac{6}{3} = \frac{14}{3}$

o **Answer:** The area is $\mathbf{14/3}$ square units.

- **Area *between* two curves:**

o To find the area between $y = f(x)$ (top) and $y = g(x)$ (bottom), you calculate: $A = \int_a^b (\text{Top Curve} - \text{Bottom Curve})dx$ $A = \int_a^b (f(x) - g(x))dx$

---

3. APPLICATION 2: REGRESSION AND CORRELATION (STATISTICS)

(This is a statistics topic, often included in Further Maths. It is not related to integration, but is a key applied maths concept).

3A. REGRESSION (LINE OF BEST FIT)

- **Definition: Linear Regression** is a statistical method to find the **line of best fit** for a set of data points (x, y).
- **Equation:** The line is $\mathbf{y = a + bx}$, where:
  - y is the dependent variable (what we want to predict).
  - x is the independent variable.
  - b is the gradient (slope) of the line.
  - a is the y-intercept.

- **Formulas (Least Squares Method):**
  - $\mathbf{b} = \frac{n\sum xy - (\sum x)(\sum y)}{n\sum x^2 - (\sum x)^2}$
  - $\mathbf{a} = \bar{y} - b\bar{x}$ (where $\bar{y} = \frac{\sum y}{n}$ and $\bar{x} = \frac{\sum x}{n}$)

3B. CORRELATION COEFFICIENT (r)

- **Definition:** The **Pearson Correlation Coefficient (r)** is a number between **-1 and +1** that measures the *strength* and *direction* of a **linear** relationship.
- **Interpretation:**
  - $r = +1$: Perfect positive linear correlation.
  - $r = -1$: Perfect negative linear correlation.
  - $r = 0$: No linear correlation.
  - $r = 0.8$: Strong positive correlation.
  - $r = 0.2$: Weak positive correlation.
- **Formula:** $r = \frac{n\sum xy - (\sum x)(\sum y)}{\sqrt{[n\sum x^2 - (\sum x)^2][n\sum y^2 - (\sum y)^2]}}$
  - (Note: The numerator is the *same* as in the formula for b).
- **Worked Example 3:** A student's scores in Maths (x) and Physics (y) are:
  - $(x, y) = \{(2,4), (3,7), (5,8), (6,11)\}$
  - Find the correlation coefficient r.

a. **Create a table and find sums ($n = 4$):**

x	y	x^2	y^2	xy
2	4	4	16	8
3	7	9	49	21
5	8	25	64	40
6	11	36	121	66
$\sum x = 16$	$\sum y = 30$	$\sum x^2 = 74$	$\sum y^2 = 250$	$\sum xy = 135$

b. **Calculate the three main parts:**

- Numerator: $n\sum xy - (\sum x)(\sum y) = 4(135) - (16)(30) = 540 - 480 = 60$.
- x-Denominator: $n\sum x^2 - (\sum x)^2 = 4(74) - (16)^2 = 296 - 256 = 40$.
- y-Denominator: $n\sum y^2 - (\sum y)^2 = 4(250) - (30)^2 = 1000 - 900 = 100$.

c. **Substitute into r formula:**

- $r = \frac{60}{\sqrt{[40][100]}} = \frac{60}{\sqrt{4000}} = \frac{60}{63.24} \approx 0.9487$.

- **Answer:** $r \approx 0.95$, indicating a very strong positive correlation.

EVALUATION

1. What is the **Fundamental Theorem of Calculus** (for definite integrals)?
2. Evaluate $\int_0^2 (x^3 + 1)dx$.
3. Find the area under the curve $y = 3x^2$ from $x = 1$ to $x = 3$.
4. What does a correlation coefficient $r = -0.9$ mean?
5. What is the formula for the gradient b of the regression line $y = a + bx$?

ASSIGNMENT

1. Evaluate $\int_1^4 (2x + \frac{1}{\sqrt{x}})dx$.
2. Find the area of the region enclosed by the curve $y = 4 - x^2$ and the x-axis. (Hint: First find where the curve cuts the x-axis to get your limits a and b).
3. The following table shows the marks of 5 students in F.Maths (x) and Physics (y).

Student	A	B	C	D	E
F.Maths (x)	10	8	12	5	15
Physics (y)	12	7	10	6	15

- (a) Calculate the correlation coefficient r.
- (b) Find the equation of the regression line y on x.

PAWILLS CONSULTING

- (c) Use your line to predict the Physics score of a student who scored 14 in F.Maths.

4. Find the area enclosed between the curves $y = x^2$ and $y = 2x$.
5. Define: Definite Integral, Indefinite Integral, Area Under Curve, Regression, Correlation.

PAWILLS CONSULTING